Classification of Smooth Manifold Structures on $\mathbb{C}P^n \times D^k$

MatFyz CONNECTIONS 2025

Smooth Manifold Structures

A manifold structure on $(\mathbb{C}P^n \times D^k, \mathbb{C}P^n \times S^{k-1})$ is a homotopy equivalence

$$(f, \partial f) : (M, \partial M) \xrightarrow{\simeq} (\mathbb{C}P^n \times D^k, \mathbb{C}P^n \times S^{k-1})$$

where M is a smooth manifold and ∂f is a diffeomorphism.

The (simple) Structure Set

$\mathcal{S}_\partial^{DIFF}(\mathbb{C}P^n \times D^k)$:= equivalence classes of smooth manifold structures, $(f_1, \partial f_1) \sim (f_2, \partial f_2)$ if

$$\begin{tikzcd}
(M_1, \partial M_1) \arrow[dr, "(f_1,\partial f_1)"] & \\
& (\mathbb{C}P^n \times D^k, \mathbb{C}P^n \times S^{k-1}) \\
(M_2, \partial M_2) \arrow[uu, "g", "\cong"'] \arrow[ur, "(f_2,\partial f_2)"'] &
\end{tikzcd}$$

Surgery Exact Sequence

- for $2n + k$ odd:

$$\mathbb{Z} \text{ or } \mathbb{Z}_2 \to \mathcal{S}_\partial(\mathbb{C}P^n \times D^k) \to \mathcal{N}_\partial(\mathbb{C}P^n \times D^k) \to 0$$

- for $2n + k \equiv 0 \mod 4$:

$$0 \to \mathcal{S}_\partial(\mathbb{C}P^n \times D^k) \to \mathcal{N}_\partial(\mathbb{C}P^n \times D^k) \xrightarrow{\sigma_{4s}} \mathbb{Z}$$

- for $2n + k \equiv 2 \mod 4$:

$$0 \to \mathcal{S}_\partial(\mathbb{C}P^n \times D^k) \to \mathcal{N}_\partial(\mathbb{C}P^n \times D^k) \xrightarrow{\sigma_{4s+2}} \mathbb{Z}_2$$

- $\mathcal{N}_\partial(\mathbb{C}P^n \times D^k)$:= set of normal invariants $\cong [Th(\mathbb{C}P^n \times D^k), G/O]$ - a generalized cohomology theory
- σ_n := the surgery obstruction map
  $\sigma_{4s}([M, f]) = \frac{1}{8}(\text{sign}(M) - \text{sign}(\mathbb{C}P^n \times D^k))$
  $\sigma_{4s+2}([M, f]) =$ Arf invariant

Normal Invariants of $\mathbb{C}P^n \times D^k$

From fibration $G/O \to BO \to BG$ we get:
$0 \to [Th(\mathbb{C}P^n \times D^k), G] \to \mathcal{N}_\partial(\mathbb{C}P^n \times D^k) \to$
$\to \widetilde{KO}^0(Th(\mathbb{C}P^n \times D^k)) \xrightarrow{J_{Th}} [Th(\mathbb{C}P^n \times D^k), BG]$.
Since $Th(X \times D^k) \simeq \Sigma^k X \vee S^k$:

$$\begin{aligned}
\mathcal{N}_\partial(\mathbb{C}P^n \times D^k) &\cong [\Sigma^k \mathbb{C}P^n, G/O] \oplus \pi_k(G/O) \\
&\cong \ker J_{\Sigma^k \mathbb{C}P^n} \oplus [\Sigma^k \mathbb{C}P^n, G] \oplus \pi_k(G/O) \\
&\cong \mathbb{Z}^t \oplus \text{torsion} \oplus \pi_k(G/O), \quad t \sim \lfloor \tfrac{n}{2} \rfloor
\end{aligned}$$

- $\ker J$: Adams conjecture for a general space X

  $$k^e(\Psi^k - \mathrm{id})\xi \in \ker J$$

  Adams, Walker: For $\mathbb{C}P^n$ (almost) equivalent to $\xi \in \widetilde{KO}^0(\mathbb{C}P^n)$ satisfying

  $$sh\,\xi = ch(1 + \beta)$$

  for some $\beta \in \widetilde{KO}^0(\mathbb{C}P^n)$
  Also holds for $\Sigma^k \mathbb{C}P^n$
- $[\Sigma^k \mathbb{C}P^n, G]$: more difficult, spectral sequence, involves π_n^s

The Surgery Obstruction Map

$\sigma_{n,k} : \mathcal{N}_\partial(\mathbb{C}P^n \times D^{2k}) \to \mathbb{Z}$ or $\mathbb{Z}_2$

- $n + k$ even:

  $$\sigma_{n,k}([M, f]) = \frac{\left(\text{sign}(M) - \text{sign}(\mathbb{C}P^n \times D^{2k})\right)}{8} \in \mathbb{Z}$$

  Gluing $(\mathbb{C}P^n \times D^{2k}, \mathrm{id})$ along boundary
  Closed manifold $\to$ Hirzebruch Signature Thm

  $$\sigma_{n,k}(\xi) = \langle \mathcal{L}(\tau_{\mathbb{C}P^n})(\mathcal{L}^{-1}(\xi) - 1), [\mathbb{C}P^n \times D^{2k}] \rangle$$

  for $\xi \in [Th(\mathbb{C}P^n \times D^{2k}), G/O]$
  In particular $\sigma_{n,k}([\Sigma^k \mathbb{C}P^n, G]) = 0$
- $n + k$ odd:
  $\sigma_{4k+2}([M, f]) = Arf(q_{M,f}) \in \mathbb{Z}_2$, where $q_{M,f}$ is the associated quadratic form
  Sullivan, Rourke:

  $$\sigma_{4k+2}([M, f]) = \langle V^2(M) \cup f^* \widetilde{K}, [\mathbb{C}P^n \times D^{2k}] \rangle$$

  for the surgery class $\widetilde{K} \in \widetilde{H}^{4*+2}(G/O; \mathbb{Z}_2)$

n	1	2	3
$\mathcal{N}_\partial(\mathbb{C}P^n \times D^2)$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^4$
n	4	5	6
$\mathcal{N}_\partial(\mathbb{C}P^n \times D^2)$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_3$	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^3$

Table 2. Normal invariants for $k = 2$

Comparison with Topological Surgery

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathcal{S}_\partial^{DIFF}(\mathbb{C}P^n \times D^{2k}) \arrow[r, "i_*"] \arrow[d] & \mathcal{S}_\partial^{TOP}(\mathbb{C}P^n \times D^{2k}) \arrow[d] \\
\mathcal{N}_\partial^{DIFF}(\mathbb{C}P^n \times D^{2k}) \arrow[r] \arrow[d, "\sigma_{n,k}^{DIFF}"] & \mathcal{N}_\partial^{TOP}(\mathbb{C}P^n \times D^{2k}) \arrow[d, "\sigma_{n,k}^{TOP}"] \\
\mathbb{Z} \text{ or } \mathbb{Z}_2 & \mathbb{Z} \text{ or } \mathbb{Z}_2
\end{tikzcd}$$

where $\mathcal{N}_\partial^{TOP}(\mathbb{C}P^n \times D^{2k}) \cong \mathbb{Z}^a \oplus \mathbb{Z}_2^b$ and $\mathcal{S}_\partial^{TOP}(\mathbb{C}P^n \times D^{2k}) \cong \mathbb{Z}^{a-1} \oplus \mathbb{Z}_2^b$ or $\mathbb{Z}^a \oplus \mathbb{Z}_2^{b-1}$
We computed $\mathrm{Im}(i_*)$ for $k = 2, 4, 6$ and $1 \leq n \leq 6$.

Further Research

$$\begin{tikzcd}
& \mathcal{S}_\partial^{DIFF}(\mathbb{C}P^n) \arrow[rr, "\eta_{\mathbb{C}P^n}^{DIFF}"] \arrow[dl, "t_{n,m}"] \arrow[dd] & & \mathcal{N}_\partial^{DIFF}(\mathbb{C}P^n) \arrow[dl, "q_n^*"] \arrow[dd] \\
\mathcal{S}_\partial^{DIFF}(L_m^{2n+1}) \arrow[rr, "\eta_{L^{2n+1}}^{DIFF}"] \arrow[dd] & & \mathcal{N}_\partial^{DIFF}(L_m^{2n+1}) \arrow[dd] & \\
& \mathcal{S}_\partial^{TOP}(\mathbb{C}P^n) \arrow[rr, "\eta_{\mathbb{C}P^n}^{TOP}"] \arrow[dl, "t_{n,m}"] & & \mathcal{N}_\partial^{TOP}(\mathbb{C}P^n) \arrow[dl, "q_n^*"] \\
\mathcal{S}_\partial^{TOP}(L_m^{2n+1}) \arrow[rr, "\eta_{L^{2n+1}}^{TOP}"] & & \mathcal{N}_\partial^{TOP}(L_m^{2n+1}) &
\end{tikzcd}$$

References

[1] G. Brumfiel.
Homotopy equivalences of almost smooth manifolds.
Commentarii Mathematici Helvetici, 46:381–407, 1971.

Smooth normal invariants	Splitting invariants
$\mathcal{N}_\partial^{DIFF}(\mathbb{C}P^6 \times D^2) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2^3$ $(m, n, p, \alpha_1, \alpha_2, \tau)$	$\mathrm{Im}(i_*) \subseteq \mathcal{N}_\partial^{TOP}(\mathbb{C}P^6 \times D^2) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2^4$ $(-m, 16m + 28n, -662m - 632n - 496p, \tau \mod 2,$ $\alpha_1 + \tau \mod 2, m + \alpha_1 + \tau \mod 2, m + \alpha_1 + \tau \mod 2)$

Table 1. Splitting invariants

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