Topology of manifolds

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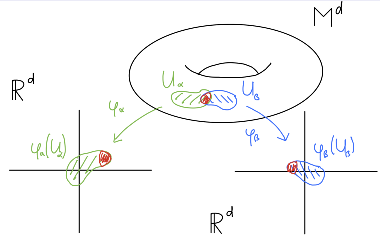
Smooth and topological manifolds



DIFF: $(M, \mathcal{F}_M)$, $\mathcal{F}_M = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ s.t. $\varphi_\beta \circ \varphi_\alpha^{-1} \in C^\infty$

TOP: $(M, \mathcal{F}_M)$, $\mathcal{F}_M = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ s.t. $\varphi_\beta \circ \varphi_\alpha^{-1} \in C^0$

Examples: S^n, $S^m \times S^n$, T^n, $F_g^\pm$, $SO(n)$, $SU(n)$, alg. geometry, ...

Homeomorphism $h\colon M \xrightarrow{\cong_{\mathrm{TOP}}} N$

Diffeomorphism $h\colon M \xrightarrow{\cong_{\mathrm{DIFF}}} N$ is $\cong_{\mathrm{TOP}}$ compatible with $\mathcal{F}_M$ and $\mathcal{F}_N$

Algebraic Topology I

The fundamental group

$$\pi_1(X) := \{c\colon [0,1] \to X \mid c(0) = c(1) = x_0\}/\simeq$$

Homotopy groups

$$\pi_k(X) := \{c\colon [0,1]^k \to X \mid c(\partial([0,1]^k) = x_0\}/\simeq$$

Homology groups

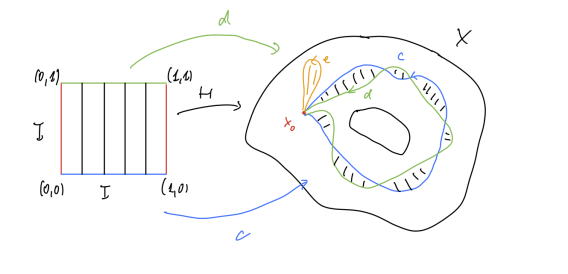
$$H_k(X) \in \mathrm{Ab} \quad \text{for all} \quad k \in \mathbb{N}$$

Functoriality

$$\begin{aligned} f\colon X \xrightarrow{\cong} Y \quad &\rightsquigarrow \quad \pi_k(f)\colon \pi_k(X) \xrightarrow{\cong} \pi_k(Y) \\ &\rightsquigarrow \quad H_k(f)\colon H_k(X) \xrightarrow{\cong} H_k(Y) \end{aligned}$$

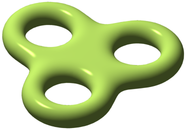
Algebraic Topology II

The fundamental group



Calculations

$\pi_1(S^1) \cong \mathbb{Z}$, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$, $H_1(F_g^+) \cong \mathbb{Z}^{2g}$



Algebraic topology III

Homotopy equivalence

$f\colon X \xrightarrow{\simeq} Y$ if $\exists\, g\colon Y \to X$ s.t. $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$

Homotopy equivalence and classical invariants

$$f\colon X \xrightarrow{\simeq} Y \Leftrightarrow \pi_k(f)\colon \pi_k(X) \xrightarrow{\cong} \pi_k(Y)$$

Hierarchy

$$\begin{array}{ccccc} \cong_{\mathrm{DIFF}} & \Rightarrow & \cong_{\mathrm{TOP}} & \Rightarrow & \simeq \\ \simeq & \not\Rightarrow & \cong_{\mathrm{TOP}} & \not\Rightarrow & \cong_{\mathrm{DIFF}} \end{array}$$

Examples

Milnor's exotic spheres $\Sigma^7 \ncong_{\mathrm{DIFF}} S^7$, but $\Sigma^7 \cong_{\mathrm{TOP}} S^7$

$\exists\; M \simeq S^{4k} \times S^n$, but $M \ncong_{\mathrm{TOP}} S^{4k} \times S^n$

$L(7;1,1) \simeq L(7;1,2)$, but $L(7;1,1) \ncong_{\mathrm{TOP}} L(7;1,2)$

The surgery structure set

Surgery theory

The structure set of X (with $\mathrm{CAT} = \mathrm{DIFF}, \mathrm{TOP}$)

$$\mathcal{S}^{\mathrm{CAT}}(X) := \{f \colon M \xrightarrow{\simeq} X\}/\sim$$

$$\text{where } (f_0 \colon M_0 \to X) \ \sim \ (f_1 \colon M_1 \to X)$$

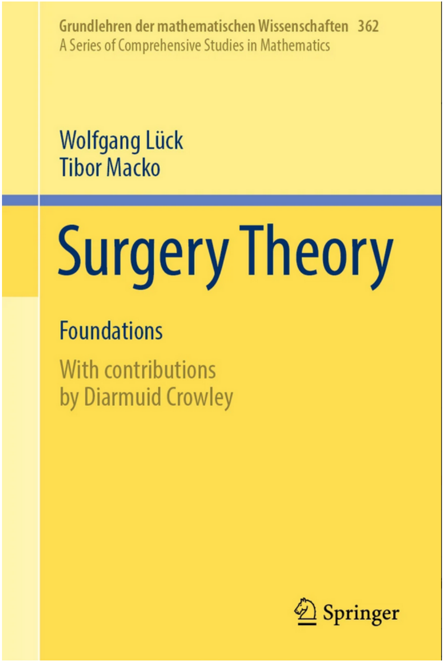
$$\text{if } \exists\ h \colon M_0 \xrightarrow{\cong_{\mathrm{CAT}}} M_1 \text{ s.t. } f_1 \circ h \simeq f_0$$

Examples

$\mathcal{S}^{\mathrm{TOP}}(S^n) \cong \{\mathrm{id}_{S^n}\} \Leftarrow$ the generalized Poincaré conjecture

$\mathcal{S}^{\mathrm{DIFF}}(S^7) \not\cong \{\mathrm{id}_{S^7}\}$

$\mathcal{S}^{\mathrm{TOP}}(S^{4k} \times S^n) \not\cong \{\mathrm{id}_{S^{4k} \times S^n}\}$



Wolfgang Lück
University of Bonn
C3-Professor of mathematics
PhD Göttingen 1984
Lexington, Mainz, Münster
Leibniz Prize
ERC Advanced grant

Surgery theory

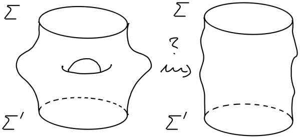
Theorem 11.22 (The surgery exact sequence, with CAT $=$ DIFF, TOP)
For an n-dimensional CAT-manifold X with $n \geq 5$ and $\pi = \pi_1(X)$

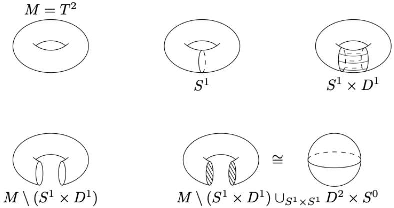
$$\cdots \to \mathcal{N}_\partial^{\mathrm{CAT}}(X \times I) \xrightarrow{\sigma_{n+1}} L_{n+1}(\mathbb{Z}\pi) \xrightarrow{\rho_n}$$
$$\xrightarrow{\rho_n} \mathcal{S}^{\mathrm{CAT}}(X) \xrightarrow{\eta_n} \mathcal{N}^{\mathrm{CAT}}(X) \xrightarrow{\sigma_n} L_n(\mathbb{Z}\pi).$$

due to Browder-Novikov-Sullivan-Wall, where

- $\mathcal{N}^{\mathrm{CAT}}(X)$ - normal cobordism - generalized cohomology theory
- $L_n(\mathbb{Z}\pi)$ - Witt group of (automorphisms of) quadratic forms, $L_n(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}/2, 0$
- σ_{n+1}, σ_n - surgery obstruction maps
- if $\pi = 0$, $\sigma_{4k}((f, \overline{f})\colon M \to X) = (1/8)(\mathbf{sign}(M) - \mathbf{sign}(X))$

Surgery





$$q\colon S^k \times D^{n-k} \hookrightarrow M$$

$$M' := D^{k+1} \times S^{n-k-1} \cup_{\mathrm{im}(q|_{S^k \times S^{n-k-1}})} (M - (\mathrm{im}(q)))\,.$$

$$\partial(D^{k+1} \times S^{n-k-1}) = S^k \times S^{n-k-1} = \partial(S^k \times D^{n-k})$$

Calculations

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\mathcal{S}^{\mathrm{DIFF}}(S^n)\rvert$	1	1	1	1	1	28	2	8	6	992	1	3	2	16256

CAT = TOP (Chapters 18,19)

$\mathcal{S}^{\mathrm{TOP}}(X)$ for

$$X = S^k \times S^l, \mathbb{C}P^n, \mathbb{R}P^n, L_N^{2d-1}, T^n, \ldots$$

For $\pi_1(X)$ finite: use representation theory.

For L_N^{2d-1} when N is odd: in terms of splitting invariants and ρ-invariants.

The Borel conjecture (Chapter 19)

$$\mathcal{S}^{\mathrm{TOP}}(BG) \cong \{\mathrm{id}_{BG}\}?$$

Via asmb: $H_n(X; \mathbf{L}_\bullet\langle 1\rangle) \to L_n(\mathbb{Z}\pi)$ using algebraic surgery (Chapter 15).

Fiber bundles

Fiber bundle

$$F \to E \xrightarrow{\pi} B \qquad \text{(in general } E \ncong B \times F\text{)}$$

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\
& U &
\end{tikzcd}$$

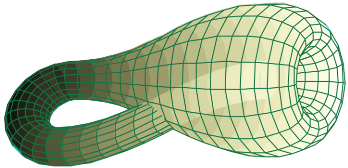
Sphere bundles over spheres $S^p \to E \to S^q$

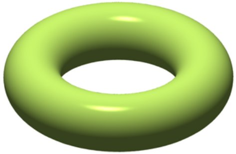
$$c_\pi \colon S^{q-1} \to SO(p+1)$$

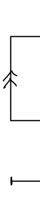
$$\begin{aligned} c \colon S^{q-1} \times S^p &\to S^{q-1} \times S^p \\ (x,y) &\mapsto (x, c_\pi(x)(y)) \end{aligned}$$

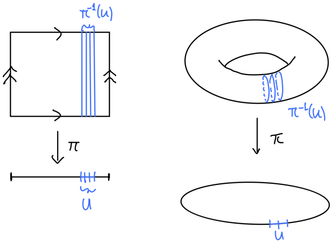
$$E := (D^q_+ \times S^p) \cup_c (D^q_- \times S^p) \xrightarrow{\pi} S^q = D^q_+ \cup_{S^{q-1}} D^q_-$$

Fiber bundles









$$S^1 \to K \xrightarrow{\pi} S^1 \qquad K \not\cong S^1 \times S^1 \qquad S^1 \to T^2 \xrightarrow{\pi} S^1 \qquad T^2 \cong S^1 \times S^1$$

Ajay Raj's thesis

Goal

$\mathcal{S}^{\mathrm{CAT}}(X)$ when $X = E$ for $S^p \to E \to S^q$

$$S^p \to E \to S^q \leftrightarrow \pi_{q-1}(SO(p+1))$$
$$\pi_{4k-1}(SO(4k)) \cong \mathbb{Z} \oplus \mathbb{Z}$$
$$(m,n) \rightsquigarrow E = M^k_{m,n} \quad \dim(M^k_{m,n}) = 8k-1$$

Theorem A

$$\mathcal{S}^{\mathrm{TOP}}(M^k_{m,n}) \cong \mathbb{Z}/2n \quad (k \geq 3, n \neq 0)$$

$F^k_{m,n} \colon \mathcal{S}^{\mathrm{DIFF}}(M^k_{m,n}) \to \mathcal{S}^{\mathrm{TOP}}(M^k_{m,n})$

Theorem B

$$\mathrm{im}(F^k_{m,n}) = \gcd(2n, \theta_k) \cdot \mathbb{Z}/2n \quad (k \geq 3, n \neq 0)$$

with $\theta_k = \mathrm{num}(B_k/4k) a_k 2^{2k-1}(2^{2k-1} - 1)$ where $a_k = 1, 2$.

Proofs I

Theorem A: follows directly from surgery exact sequence

$$\mathcal{S}^{\mathrm{TOP}}(M^k_{m,n}) \cong \mathcal{N}^{\mathrm{TOP}}(M^k_{m,n})$$

Theorem B: Construction of examples using htpy:

$$\begin{tikzcd}
\pi_{4k-1}(SO(4k)) \arrow[r, "\cong"] \arrow[d, "J_{4k-1}"'] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "(J,\mathrm{id})"] \\
\pi_{4k-1}(SH(4k)) \arrow[r, "\cong"] & \pi^S_{4k-1} \oplus \mathbb{Z}
\end{tikzcd}$$

$$\rightsquigarrow f^k_t \colon M^k_{m+j_k t,n} \xrightarrow{\simeq} M^k_{m,n} \quad (t \in \mathbb{Z})$$

Here $j_k = |\mathrm{im} J|$ known for $J \colon \pi_{4k-1}(SO) \to \pi^S_{4k-1}$ by work of Adams in terms of Bernoulli numbers, $j_k = \mathrm{denom}(B_k/4k)$.

Proposition: $[f^k_t] = \theta_k \cdot t \in \mathcal{S}^{\mathrm{TOP}}(M^k_{m,n}) \cong \mathbb{Z}/2n$.

Proofs II

James-Whitehead:

$$E/S^{4k-1} \simeq_s S^{4k} \vee S^{8k-1}$$

Determine $\mathcal{N}^{DIFF}(E) \cong [E, G/O] \to \mathcal{N}^{TOP}(E) \cong [E, G/TOP]$ via

$$\begin{tikzcd}
{[S^{4k}, G/O]} \arrow[r] \arrow[d] & {[S^{4k}, G/TOP]} \arrow[d] \\
{[S^{4k} \vee S^{8k-1}, G/O]} \arrow[r] \arrow[d] & {[S^{4k} \vee S^{8k-1}, G/TOP]} \arrow[d] \\
{[E, G/O]} \arrow[r, "\chi"] \arrow[d] & {[E, G/TOP]} \arrow[d] \\
{[S^{4k-1}, G/O]} \arrow[r] & {[S^{4k-1}, G/TOP]}
\end{tikzcd}$$

using known knowledge (Brumfiel) about $\pi_i(G/O) \to \pi_i(G/TOP)$ from the Kervaire-Milnor braid. Finalize via $\mathcal{S}^{\mathrm{TOP}}(E) \cong \mathcal{N}^{TOP}(E)$.

Proofs III

The Kervaire-Milnor braid ($\Theta_n = \mathcal{S}^{\mathrm{DIFF}}(S^n)$):

$$\begin{tikzcd}
\pi_n(G) \arrow[dr] \arrow[rr, bend left] & & \pi_n(G/TOP) \arrow[dr] \arrow[rr, bend left] & & \Theta_{n-1} \\
& \pi_n(G/O) \arrow[ur] \arrow[dr] & & \pi_{n-1}(TOP) \arrow[ur] \arrow[dr] & \\
\Theta_n \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(O) \arrow[ur] \arrow[rr, bend right] & & \pi_{n-1}(G)
\end{tikzcd}$$

Related information

Other Phd students

Serhii Dylda: [min. 2022]
algebraic surgery via ball complexes
studies product formulas in $\mathcal{S}^{\mathrm{TOP}}(X \times Y) \cong \mathbb{S}_*(X \times Y)$
Samuel Kalužný: [min. 2024]
studies $\mathcal{S}^{\mathrm{DIFF}}(\mathbb{C}P^n \times D^k) \to \mathcal{S}^{\mathrm{TOP}}(\mathbb{C}P^n \times D^k)$
Marián Poturnay:
studies $\mathcal{N}^{\mathrm{TOP}}(X)$ localized away from 2 via Conner-Floyd isomorphism with $KO(X)$

Other interests

Topological data analysis
Topological deep learning
Formal proofs (LEAN)