Innovative Approaches to Teaching Probability in Secondary Schools

MatFyz CONNECTIONS 2025

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
Univerzita Komenského v Bratislave

MATFYZ CONNECTIONS

Michaela Vargová
Peter Vankúš
Comenius University Bratislava, Slovakia

Misconceptions in Probability Learning

- Misconceptions occur even in basic probability concepts at low levels of abstraction (Batanero & Sánchez, 2005).
- Probabilistic reasoning often leads to counterintuitive results already at an elementary level (Borovcnik & Peard, 1996).
- In most areas of mathematics, counterintuitive results appear only at high levels of abstraction.
- This special nature of probability explains why misconceptions and difficulties persist even in secondary school (Shaughnessy, 1992; Fischbein, Nello, & Marino, 1991).

- Batanero, C., & Sánchez, E. (2005). What is the nature of high school students' conceptions and misconceptions about probability?
- Borovcnik, M., & Peard, R. (1996). Probability.
- Shaughnessy, J. M. (1992). Research in probability and statistics.
- Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgments in children and adolescents.

Historical Misconception: d'Alembert's Error

- Even great mathematicians can make mistakes in probability (Gorroochurn, 2014).
- Jean le Rond d'Alembert (1717–1783) analyzed the problem:

  "What is the probability that at least one head appears in two tosses of a fair coin?"
- He rejected the correct answer 3/4 and reasoned incorrectly:
  - If the first toss is heads, no second toss is needed.
  - Considered only outcomes: H, TH, TT.
  - Concluded probability = 2/3.
- His reasoning was flawed because these outcomes are not equally likely.
- The incorrect solution was even published in his Encyclopédie article (d'Alembert, 1754, Vol. IV, pp. 512–513).

- Gorroochurn, P. (2014). Errors of probability in historical context.
- d'Alembert, J. L. R. (1754). Croix ou pile. In D. Diderot & J. L. R. d'Alembert (Eds.), Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers

Addressing Misconceptions in Probability

- Effective tools: Experimentation, manipulatives, and visualization (Paliwal, 2018).
- Gradual development of concepts aligns with the constructivist approach to teaching.
- After the constructivist phase, it is important to:
    - Formalize the concepts.
    - Introduce correct mathematical symbolism and terminology.

- Paliwal, V. (2018). Do manipulatives foster pre-service teachers' understanding of probability?

Innovative approaches to teaching probability

- Developed as part of the KEGA 037UK-4/2024 *Innovative learning technologies in the preparation of future mathematics teachers* and *Digital Transformation of Education and Schools* (DiTEdu, code ITMS2014+: 401402DVR6).
- Aim to foster a deep conceptual understanding of probabilistic concepts through their gradual constructivist development, with an emphasis on visualization, manipulative activities, and experimentation.
- The topics covered include **geometric probability, conditional probability and expected value.**

DiTEdu
Digitálna transformácia vzdelávania a školy
Financované Európskou úniou
ncDTV
NIVAM
NÁRODNÝ INŠTITÚT VZDELÁVANIA A MLÁDEŽE
UNIVERZITA PAVLA JOZEFA ŠAFÁRIKA V KOŠICIACH
UNIVERZITA KOMENSKÉHO V BRATISLAVE
MINISTRY OF EDUCATION, RESEARCH, DEVELOPMENT AND YOUTH OF THE SLOVAK REPUBLIC
Kultúrna a edukačná grantová agentúra MŠVVaM SR (KEGA)

- Educational materials

- Feedback

- Future Work

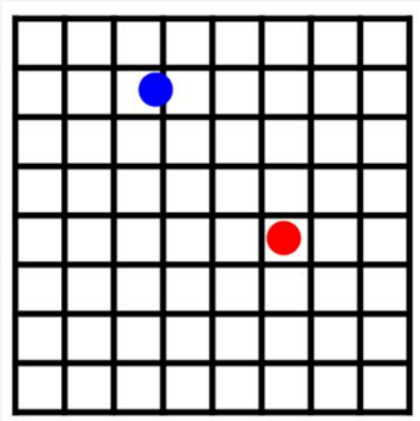
Educational materials

Geometric probability: Franc-carreau (Fair-square)

We toss a coin into the air above a square-grid game board.

We win if the coin lands entirely inside one of the squares. If the coin intersects any side of a square, we lose.

What is the probability that the coin will lie completely inside a square (i.e., that it will not land on any of the square's sides)?



(Kubáček, 2010)

- Kubáček, Z. (2010). Matematika pre 2. ročník gymnázií, 2. časť. Orbis Pictus Istropolitana.

Geometric probability: misconceptions

- **Misuse of Discrete Reasoning**
  - Students list discrete outcomes as if using Laplace's classical definition.
  - They fail to see that the situation is fundamentally different from problems with finite sets of equally likely outcomes.
- **Inappropriate "Target-Shooting" Example**
  - Even university students propose shooting at a target as an introductory task.
  - This is not a random process because the shooter deliberately aims.
  - Indicates that students do not understand the assumptions required for geometric probability.
- **Confusion Between Impossible Events and Probability Zero**
  - Task: Choose $x, y \in (0,1)$ uniformly and independently; find $P(x + y = 3/2)$.
  - Students confuse an impossible event with an event of probability zero.

Conditional probability: Visualisations in *Cabs' Problem*

A cab was involved in a hit and run accident at night. Two cab companies, the Green and the Blue, operate in the city. 85% of the cabs in the city are Green and 15% are Blue.

A witness identified the cab as Blue. The court tested the reliability of the witness under the same circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 75% of the time and failed 25% of the time.

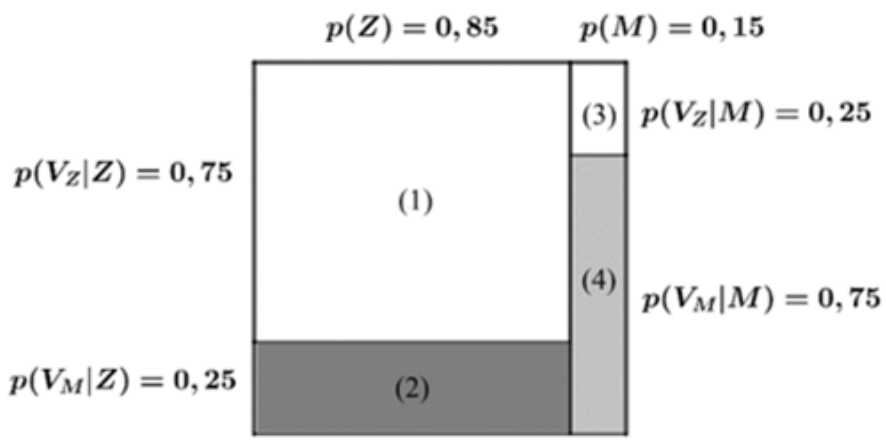
What is the probability that the cab involved in the accident was Blue rather than Green knowing that this witness identified it as Blue?

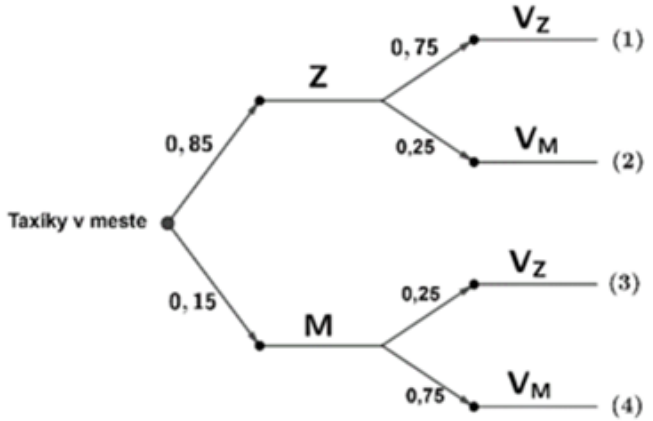
(Tversky & Kahneman, 1980)

- Tversky, A., & Kahneman, D. (1980). Causal schemas in judgments under uncertainty.

Visualizations in *Cabs' Problem*

	V_Z	V_M	∑
Z	$\|Z \cap V_Z\| =$ $= 0{,}75 \cdot 8\,500 = 6\,375$	$\|Z \cap V_M\| =$ $= 0{,}25 \cdot 8\,500 = 2\,125$	8 500
M	$\|M \cap V_Z\| =$ $= 0{,}25 \cdot 1\,500 = 375$	$\|M \cap V_M\| =$ $= 0{,}75 \cdot 1\,500 = 1\,125$	1 500
∑	6 750	3 250	$\|\Omega\| = 10\,000$





(1) $p(Z \cap V_Z)$ (2) $p(Z \cap V_M)$ (3) $p(M \cap V_Z)$ (4) $p(M \cap V_M)$

Z – muža zrazil zelený taxík; M – muža zrazil modrý taxík; V_Z – muž videl farbu taxíka, ktorý ho zrazil, ako zelenú; V_M – muž videl farbu taxíka, ktorý ho zrazil, ako modrú.

- Vankúš, P., & Vargová, M. (2025). Vizualizácie úlohy z podmienenej pravdepodobnosti.

Conditional probability: misconceptions

- **Not Recognizing the Restricted Probability Space**
  - Students fail to notice that a condition narrows the probability space.
  - This affects the frequencies used in calculating the desired probability.
- **Confusing $P(A \mid B)$ and $P(B \mid A)$**
  - Students often mix up the two conditional probabilities.
- **Uncertainty about:**
  - Which value they are supposed to find.
  - Which value is already given in the problem.

Expected value: Dices with modified numbering

The table below shows six dice and the numbering of their faces.

Try to guess which die will produce the highest total sum after 60 rolls.

Then, try to explain why you think this particular die will have the highest sum.

	Číslovanie stien
Kocka č. 1	1 – 2 – 2 – 4 – 4 – 6
Kocka č. 2	1 – 1 – 3 – 3 – 5 – 6
Kocka č. 3	1 – 1 – 3 – 4 – 4 – 6
Kocka č. 4	1 – 2 – 2 – 3 – 5 – 6
Kocka č. 5	1 – 1 – 2 – 4 – 5 – 6
Kocka č. 6	1 – 2 – 2 – 2 – 6 – 6

(Križánková, 2024)

- Križánková, M. (2024). Hrové vyučovanie pravdepodobnosti a štatistiky.

Expected value: misconceptions

- **Misunderstanding the Law of Large Numbers**
  - Students think the sum cannot be estimated without knowing exact outcomes.
- **Expected Value Must Be a Possible Outcome**
  - Students struggle with the idea that the expected value may not appear on the die.
  - Example: mean of a standard die is 3.5, which is not a face value
- **Confusing Expected Value with the Most Probable Value**
  - Some choose the die with more 6s as “best”.
  - Fail to see that dice with the same expected value are equivalent.
- **Expected Value as a Guaranteed Result**
  - Students believe a game with positive expected value must always produce a win, or that repeated play cannot lead to losses.
  - Corrected through experimentation + reminding them of the Law of Large Numbers.

Feedback

Overview of Collected Teacher Feedback

- Feedback on selected instructional materials was gathered through questionnaires.
- **Conditional probability materials**
    - Presented during a workshop at *Two Days with Mathematics Education 2024*.
- **Expected value materials**
    - Presented during a workshop at *Two Days with Mathematics Education 2025*.
- **All three topics (geometric probability, conditional probability, expected value)**
    - Included in the program *Innovative Training: Learning to Teach Mathematics that Inspires*.

Note on Geometric Probability

- Materials used during the online portion of the training.
- No feedback collected for this topic.

Feedback on the materials on expected value

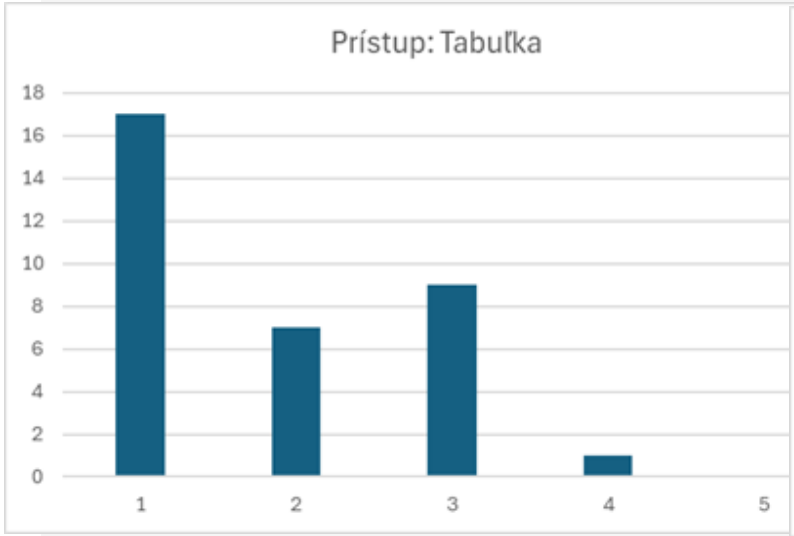
- Teachers' responses were analyzed using thematic analysis (Schreier et al., 2019).
- The table shows: Identified codes and example of teacher statements illustrating each code.

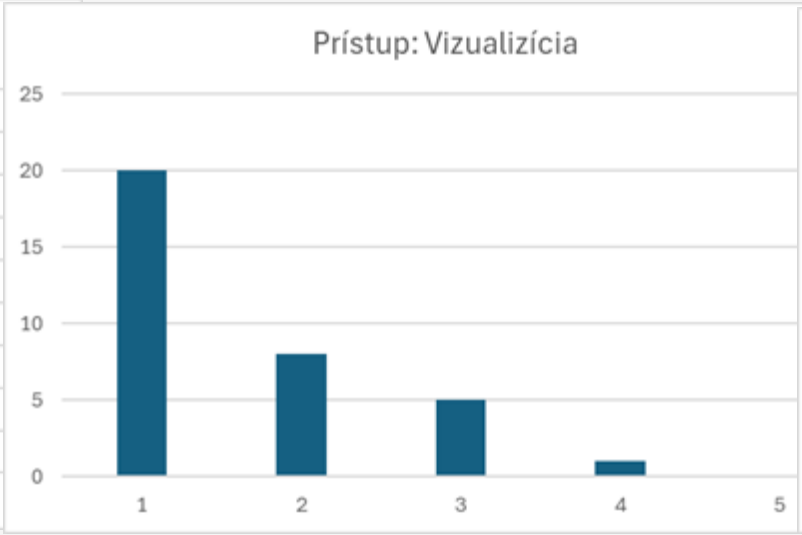
Code	Example Statements
Positive evaluation of the suitability of the task sequence	*The tasks were chosen very appropriately; I especially liked the task with the balls and assigning values to them.* *The tasks are nice; they promote creative thinking.* *Suitable, nicely connected sequence.*
Suggestion to add tasks to the methodology	*I liked them; I would add one more similar task at the end.*
Comments on task wording	*Some formulations were more difficult to understand, e.g., what is meant by "trial".* *It would be better to explain at the beginning what one trial means (60 dice rolls).*

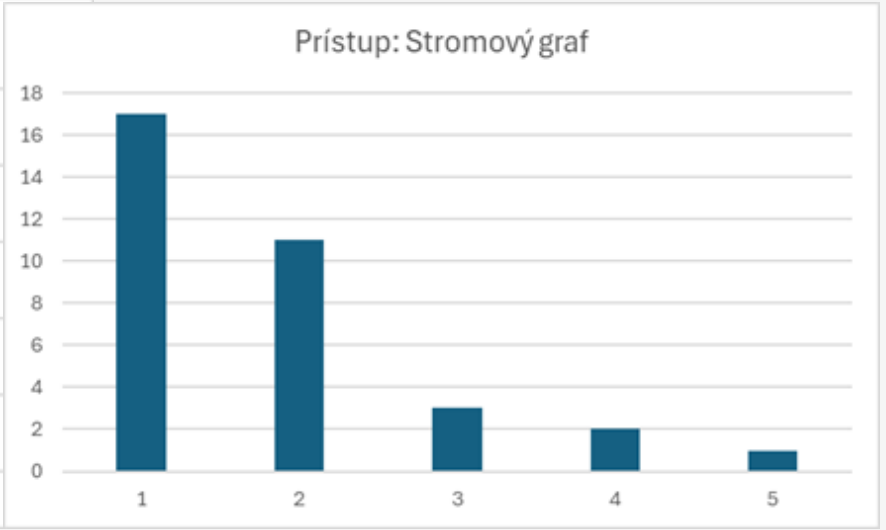
- Schreier, M, et al. (2019). Qualitative content analysis: Conceptualizations and challenges in research practice.

Feedback on the materials on conditional probability 1

- **How did you like the presented approaches to solving tasks focused on the use of conditional probability?**
- Approaches: *table of frequencies, unit-square visualization, tree diagram*; scale: positive 1—2—3—4—5 negative.

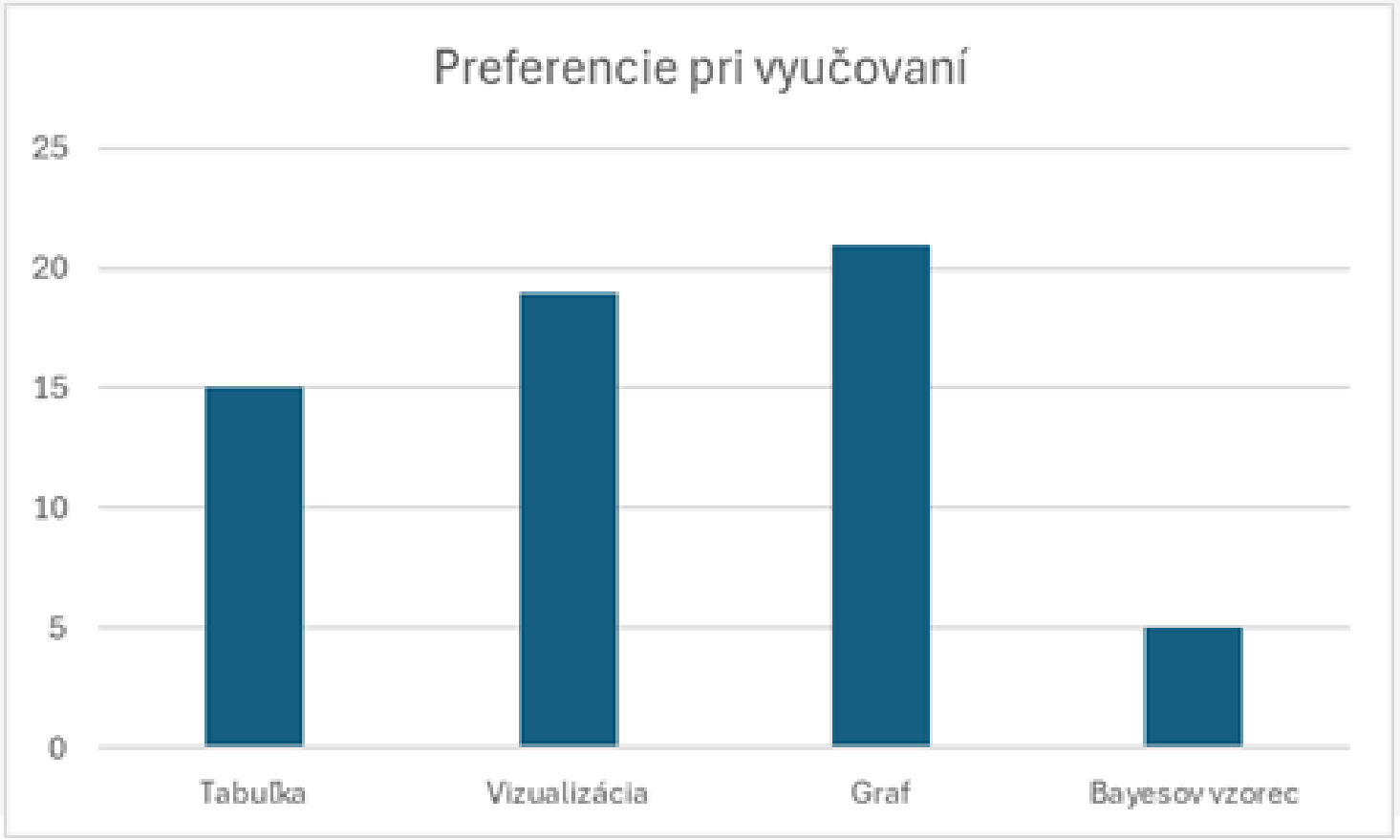






Feedback on the materials on conditional probability 2

- **Which approach to solving the tasks would you prefer when teaching your students?**
- Approaches: *a) table of frequencies, b) unit-square visualization, c) tree diagram, d) Bayes' formula*; (teachers were allowed to select multiple options).



Future Work

Future Work

- Verification of the proposed materials by in-service teachers within school mathematics teaching.
- Revision of the materials based on the collected feedback.
- Design and development of additional materials and activities related to introducing probability and statistics concepts in secondary school.

Resources

- Batanero, C., & Sánchez, E. (2005). What is the nature of high school students' conceptions and misconceptions about probability? In G. A. Jones (Ed.), Exploring probability in school: Challenges for teaching and learning (pp. 241–266). Springer. https://doi.org/10.1007/0-387-24530-8_11
- Borovcnik, M., & Peard, R. (1996). Probability. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), International handbook of mathematics education [Part 1] (pp. 239–288). Kluwer.
- d'Alembert, J. L. R. (1754). Croix ou pile. In D. Diderot & J. L. R. d'Alembert (Eds.), Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers (Vol. 4, p. 250).
- Fischbein, E., Nello, M. S., & Marino, M. S. (1991). Factors affecting probabilistic judgments in children and adolescents. Educational Studies in Mathematics, 22, 523–549.
- Gorroochurn, P. (2014). Errors of probability in historical context. In M. Pitici (Ed.), The best writing on mathematics 2013 (pp. 191–212). Princeton University Press. https://doi.org/10.1515/9781400847990-020

Resources

- Križánková, M. (2024). Hrové vyučovanie pravdepodobnosti a štatistiky. In J. Gurican & J. Tekel (Eds.), Študentská vedecká konferencia FMFI UK, Bratislava, 2024. Zborník príspevkov (s. 127–132). Fakulta matematiky, fyziky a informatiky, Univerzita Komenského.
- Kubáček, Z. (2010). Matematika pre 2. ročník gymnázií, 2. časť. Orbis Pictus Istropolitana.
- Paliwal, V. (2018). Do manipulatives foster pre-service teachers' understanding of probability? In University of West Georgia (p. 2). https://files.eric.ed.gov/fulltext/EJ1191659.pdf
- Schreier, M., Stamann, C., Janssen, M., Dahl, T., & Whittal, A. (2019). Qualitative content analysis: Conceptualizations and challenges in research practice.
- Shaughnessy, J. M. (1992). Research in probability and statistics. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 465–494). Macmillan.
- Tversky, A., & Kahneman, D. (1980). Causal schemas in judgments under uncertainty. In M. Fishbein (Ed.), Progress in social psychology (pp. 49–72). Hillsdale, NJ: Erlbaum.
- Vankúš, P., & Vargová, M. (2025). Vizualizácie úlohy z podmienenej pravdepodobnosti. Učitel Matematiky, 32(4), 245–248. https://ojs.cuni.cz/ucitel/article/view/4780

Thank you for your attention

Michaela Vargová
Peter Vankúš
Comenius University Bratislava, Slovakia

michaela.vargova@fmph.uniba.sk
peter.vankus@fmph.uniba.sk

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
Univerzita Komenského v Bratislave

MATFYZ CONNECTIONS