Fluid Dynamics in Porous Media

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Transport equations

A "profile" $F : \mathbb{R} \to \mathbb{R}$ (modeling some physical quantity) that moves at a constant speed v to the right is represented by

$$f(x,t) := F(x - vt)$$

This function of time $t \in \mathbb{R}$ and space variable $x \in \mathbb{R}$ satisfies the partial differential equation (PDE)

$$\frac{\partial f}{\partial t}(x,t) + v\frac{\partial f}{\partial x}(x,t) = 0 \qquad \text{(or just } f_t + vf_x = 0\text{)}$$

because the partial derivatives are $-vF'(x - vt)$ and $F'(x - vt)$.

In two (and more) space dimensions (i.e. $\mathbf{x} \in \mathbb{R}^2$), this becomes

$$f_t + \mathbf{v} \cdot \nabla f = 0$$

with $\mathbf{v} \in \mathbb{R}^2$ and gradient $\nabla f := (f_{x_1}, f_{x_2})$, so $\mathbf{v} \cdot \nabla f = v_1 f_{x_1} + v_2 f_{x_2}$.

The same PDE holds when the transporting velocity $\mathbf{v}(\mathbf{x}, t)$ depends on $(\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}$, and even when $\mathbf{f}$ is vector-valued.

Euler equations for ideal fluids

A particularly important case of this is when $\mathbf{f} = \mathbf{v}$, that is,

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = 0$$

(with $\nabla \mathbf{v}$ a 2×2 matrix). Here $\mathbf{v}(\mathbf{x}, t)$ is velocity of the fluid molecules at space-time location $(\mathbf{x}, t)$, which are transported by the same velocity (Burgers equation).

Problem: in this model, fluid molecules do not interact.

In fact, liquids are incompressible and molecules push against each other, causing (instantaneous) build-up of pressure, which then acts as an additional force (adding to acceleration $\mathbf{v}_t$):

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p$$

One new unknown $p(\mathbf{x}, t) \in \mathbb{R}$ requires adding one equation. Physically, this encodes incompressibility of the fluid:

$$\nabla \cdot \mathbf{v} = 0$$

with $\nabla \cdot \mathbf{v} = (v_1)_{x_1} + (v_2)_{x_2}$ the divergence of $\mathbf{v}$. This can be used to determine p, and the two PDE are the Euler equations.

Euler equations for ideal fluids

Pressure complicates things, but can be removed by applying

$$\nabla^\perp \cdot (f_1, f_2) = (-\partial_{x_2}, \partial_{x_1}) \cdot (f_1, f_2) = -(f_1)_{x_2} + (f_2)_{x_1}$$

to the first PDE. From $\nabla^\perp \cdot \nabla p = -(p_{x_1})_{x_2} + (p_{x_2})_{x_1} = 0$ we get

$$\omega_t + \mathbf{v} \cdot \nabla \omega = 0$$

with vorticity $\omega := \nabla^\perp \cdot \mathbf{v} = -(v_1)_{x_2} + (v_2)_{x_1}$ representing the amount and direction of fluid rotation (this also uses $\nabla \cdot \mathbf{v} = 0$). This is the vorticity form of Euler – a single transport PDE!

But now we need to express $\mathbf{v}$ from ω (a little complicated):

$$\mathbf{v}(\mathbf{x}, t) = \nabla^\perp \Delta^{-1} \omega(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{y} - \mathbf{x})^\perp}{|\mathbf{y} - \mathbf{x}|^2} \, \omega(\mathbf{y}, t) \, d\mathbf{y}$$

because $\nabla^\perp \cdot \nabla^\perp = \partial_{x_1 x_1} + \partial_{x_2 x_2} = \Delta$. This makes the PDE an active scalar equation ($\mathbf{v}$ depends on the transported quantity).

Main questions: do solutions exist for all time (yes) and how fast can their derivatives grow (double-exponentially in time)?

Incompressible Porous Media equation (IPM)

Another important active scalar PDE is IPM. The unknown quantity is now the (variable) density ρ of an incompressible fluid inside a porous medium (e.g. a limestone aquifer):

$$\rho_t + \mathbf{v} \cdot \nabla \rho = 0$$

Now gravity is $(0, -\rho)$ (let $g = 1$) and $\mathbf{v}$ is given by Darcy's law

$$\mathbf{v} := -\nabla p - (0, \rho) \qquad \text{(not } v_t\text{) and} \qquad \nabla \cdot \mathbf{v} = 0$$

Here $\nabla^\perp \cdot \mathbf{v} = -\rho_{x_1}$, so $\mathbf{v} = -\nabla^\perp \Delta^{-1} \rho_{x_1}$ (this is more singular).

A particularly important case is the Muskat problem, where

$$\rho(\mathbf{x}, t) = \begin{cases} \rho_0 & \mathbf{x} \in \Omega^t \\ \rho_1 & \mathbf{x} \notin \Omega^t \end{cases}$$

This models the dynamic of two fluids with densities $\rho_0 < \rho_1$ (e.g. oil and water, or fresh and salt water), where Ω^t is the region of the lighter fluid and $\partial\Omega^t$ is the fluid interface.

PDE dynamic then transports $\partial\Omega^t$ with the above velocity $\mathbf{v}$ (ρ_{x_1} being a Dirac delta measure on $\partial\Omega^t$; note that $|x|'' = 2\delta_0$).

Well-posedness/ill-posedness/overturning on $\mathbb{R}^2$

Originally formulated by (petroleum engineer) Muskat in 1937, studied extensively in the last 20 years (mainly on $\mathbb{R}^2$).

Rayleigh-Taylor stable regime when lighter fluid is everywhere above the denser one; otherwise unstable regime.

Many results proving local well-posedness in stable regime for sufficiently smooth interfaces (and global well-posedness for interfaces close to flat), incl. Siegel, Caflisch, Howison (2004); Córdoba, Gancedo (2007); Constantin, Gancedo, Shvydkoy, Vicol (2017); Cameron (2019); Alazard, Nguyen (2023);...

Ill-posedness in unstable regime: Córdoba, Gancedo (2007).

Moreover, initial stable regime interfaces can overturn in finite time and the problem becomes ill-posed (a form of singularity): Castro, Córdoba, Fefferman, Gancedo, López-Fernández (2012).

Stable regime interface singularity development still open on $\mathbb{R}^2$.

PDE for the stable regime Muskat interface on $\mathbb{R}^2$

In stable regime on $\mathbb{R}^2$, Muskat can be transformed into a PDE (or rather an integro-differential equation) for a function f whose graph is the interface (**v** only matters on it):

$$f_t(x,t) = \frac{\rho_1 - \rho_0}{2\pi} \int_{\mathbb{R}} \frac{y\,[f_x(x,t) - f_x(x-y,t)]}{y^2 + [f(x,t) - f(x-y,t)]^2} dy$$

This comes from $\mathbf{v} = -\nabla^{\perp}\Delta^{-1}\rho_{x_1}$, which shows that **v** has the same degree of regularity as ρ (discontinuous on the interface, but only tangential component, so normal velocity well-defined).

This also makes $\rho_t + \mathbf{v} \cdot \nabla\rho = 0$ supercritical (ρ is only bounded, so **v** needs to be 1 derivative smoother to be Lipschitz).

Supercriticality generally suggests finite time singularities, but gravity also works to smooth the dynamic in the stable regime.

Muskat in the stable regime on the half-plane $\mathbb{R} \times \mathbb{R}^+$

Aquifers (e.g. sand or sandstone) lie above or in-between impermeable rocky layers. If the fluid interface gets close to the bottom (or top) layer, need Muskat on $\mathbb{R} \times \mathbb{R}^+$ (or on horizontal strips if the two layers are close). This is

$$f_t(x,t) = \frac{\rho_1 - \rho_0}{2\pi} \sum_{\pm} \int_{\mathbb{R}} \frac{y\,[f_x(x,t) \pm f_x(x-y,t)]}{y^2 + [f(x,t) \pm f(x-y,t)]^2} dy \qquad (1)$$

Particularly relevant when one fluid invades a region occupied exclusively by the other (water flows underneath oil along the bottom layer, saltwater flows underneath freshwater, etc.). Then the interface even touches/lies on the boundary.

Can stable regime singularities occur on the boundary? Existing $\mathbb{R}^2$ theory required initial data to vanish at $\pm\infty$ or be periodic or small. How about more general interfaces?

Cordoba, Granero-Belinchon, Orive (2014): Muskat on a strip, interface away from boundaries (can be done on $\mathbb{R} \times \mathbb{R}^+$).

Local well-posedness and blow-up criterion on $\mathbb{R} \times \mathbb{R}^+$

We can even allow $O(|x|^{1-})$ growth of f as $|x| \to \infty$ (optimal).
But for simplicity we will only consider bounded interfaces, with uniformly bounded H^2-norms of $f_x(\cdot, t)$ on unit intervals. Define

$$\|g\|_{\tilde{L}^2(\mathbb{R})} := \sup_{x \in \mathbb{R}} \|g\|_{L^2([x-1,x+1])} \qquad \|g\|_{\tilde{H}^3(\mathbb{R})} := \sum_{k=0}^{3} \|g^{(k)}\|_{\tilde{L}^2(\mathbb{R})}$$

Theorem (Z. 2024+)

If $\psi \geq 0$ has $\|\psi\|_{\tilde{H}^3(\mathbb{R})} < \infty$, there is $T_\psi > 0$ such that:

(i) There is a unique classical solution $f \geq 0$ to (1) on $\mathbb{R} \times [0, T_\psi)$ with $f(\cdot, 0) \equiv \psi$ such that for any $T \in [0, T_\psi)$,

$$\sup_{t \in [0,T]} \|f(\cdot, t)\|_{\tilde{H}^3(\mathbb{R})} < \infty$$

(ii) If $T_\psi < \infty$, then for each $\gamma \in (0, 1]$,

$$\int_0^{T_\psi} \left(\|f_x(\cdot, t)\|^4_{L^\infty(\mathbb{R})} + \|f_{xx}(\cdot, t)\|^4_{\dot{C}^\gamma(\mathbb{R})} \right) dt = \infty$$

Finite time singularity for Muskat on $\mathbb{R} \times \mathbb{R}^+$

Here

$$\|g\|_{\dot{C}^\gamma(\mathbb{R})} := \sup_{x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\gamma}$$

Extends to Muskat on $\mathbb{R}^2$ and on horizontal strips.

Theorem (Z. 2024+)

If $\min \psi = 0$, $\psi - \psi_\infty \in H^3(\mathbb{R})$ *for some constant* $\psi_\infty \in (0, \infty)$, *and* $\|\psi'\|_{L^\infty(\mathbb{R})} \leq \frac{3}{10}$, *then* $T_\psi < \infty$ *and* f *above satisfies*

$$\|f_x\|_{L^\infty(\mathbb{R} \times [0, T_\psi))} \leq \frac{3}{10} \quad \Longrightarrow \quad \int_0^{T_\psi} \|f_{xx}(\cdot, t)\|^4_{\dot{C}^\gamma(\mathbb{R})} \, dt = \infty$$

for each $\gamma \in (0, 1]$.

This is the "denser fluid invading from both sides" scenario.

Different dynamic from ideas to obtain singularity for IPM with initial data close to unstable regime for Muskat (e.g. Kiselev, Yao (2023); with forcing: Córdoba, Martínez-Zoroa (2024+)).

Maximum principles for Muskat

The last theorem is proved via these maximum principles, with

$$\lambda(a,b,c) := \frac{\rho_1 - \rho_0}{2\pi} \frac{(a+b)^2(a-b)^2}{2[c^2+(a+b)^2]\,[c^2+(a-b)^2]} \geq 0$$

Theorem (Z. 2024+)

(i) If $\min \psi = 0$*, then* $\min f(\cdot,t) = 0$ *for each* $t \in [0, T_\psi)$.

(ii) If $\psi - \psi_\infty \in H^3(\mathbb{R})$*, then for each* $t \in [0, T_\psi)$ *we have*

$$\frac{d}{dt}\|f(\cdot,t) - \psi_\infty\|^2_{L^2(\mathbb{R})} \leq -\int_{\mathbb{R}^2} \lambda\big(f(x,t), f(x-y,t), y\big)\, dxdy$$

(iii) If $\|f_x(\cdot,t')\|_{L^\infty(\mathbb{R})} \leq \frac{3}{10}$ *for some* $t' \in [0, T_\psi)$*, then* $\|f_x(\cdot,t)\|_{L^\infty(\mathbb{R})}$ *is decreasing on* $[t', T_\psi)$.