# Double Nonlinear Diffusion Equations in a Two-Component Domain

## Mathematical model

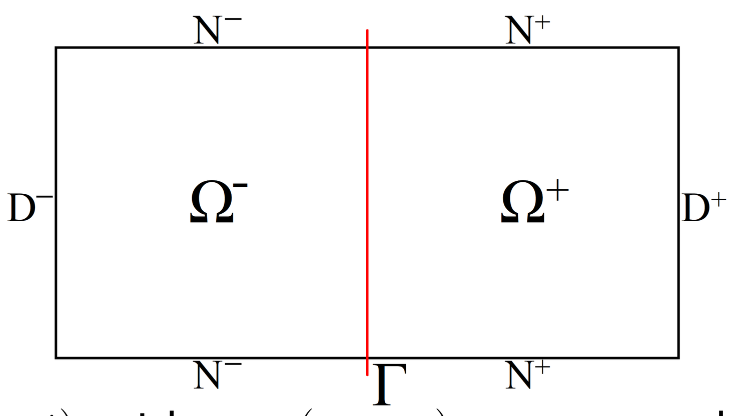
Let $0 < m < p$ and $0 < \sigma < r$ be given. Our aim is to study the following problem. Consider the equations

$$\left.\begin{aligned} \partial_t u^m - \nabla\cdot\left(|\nabla u|^{p-1}\nabla u\right) &= 0 \quad \text{in } Q^- \\ \partial_t v^\sigma - \nabla\cdot\left(|\nabla v|^{r-1}\nabla v\right) &= 0 \quad \text{in } Q^+ \end{aligned}\right\}, \tag{1}$$

where

$$Q^- = \Omega^- \times (0,T)\,,\ \ \Omega^- = (-\ell,0)\times(-\hbar,\hbar)\,,$$
$$Q^+ = \Omega^+ \times (0,T)\,,\ \ \Omega^+ = (0,\ell)\times(-\hbar,\hbar)\,,$$

for positive $\ell, \hbar, T$.




The nonnegative functions $u = u(x,t)$ and $v = v(x,t)$, with $x = (x_1, x_2)$ are assumed to satisfy the following contact conditions at $x_1 = 0$:

$$\left.\begin{aligned} \left(|\nabla u|^{p-1}\nabla u - |\nabla v|^{r-1}\nabla v\right)\cdot(1,0) &= 0 \\ v &= M u^\omega \end{aligned}\right\} \quad \text{on } S = \Gamma\times(0,T) \tag{2}$$

for given $0 < M, \omega < \infty$, where $\Gamma = \{0\}\times(-\hbar,\hbar)$. On the remaining parts of $(\partial\Omega^- \setminus \Gamma)$ and $(\partial\Omega^+ \setminus \Gamma)$ we consider the homogenous Dirichlet and Neumann boundary conditions of the form:

$$\left.\begin{aligned} u &= 0 \quad \text{on } D^- \times (0,T)\,,\ D^- = \{-\ell\}\times(-\hbar,\hbar)\,, \\ v &= 0 \quad \text{on } D^+ \times (0,T)\,,\ D^+ = \{\ell\}\times(-\hbar,\hbar) \end{aligned}\right\} \tag{3}$$

and

$$\left.\begin{aligned} |\nabla u|^{p-1}\nabla u\cdot\nu &= 0 \quad \text{on } N^- \times (0,T)\,,\ N^- = \partial\Omega^- \setminus (\Gamma\cup D^-)\,, \\ |\nabla v|^{r-1}\nabla v\cdot\nu &= 0 \quad \text{on } N^+ \times (0,T)\,,\ N^+ = \partial\Omega^+ \setminus (\Gamma\cup D^+)\,, \end{aligned}\right\} \tag{4}$$

where $\nu$ is the outward pointing unit normal vector at any point of $N^-$ and $N^+$ except the corners. For definiteness, we study our problem subject to the appropriate initial conditions

$$\left.\begin{aligned} u(\,\cdot\,,0) &= u_0 \quad \text{on } \Omega^- \\ v(\,\cdot\,,0) &= v_0 \quad \text{on } \Omega^+ \end{aligned}\right\} \tag{5}$$

for given bounded nonnegative functions $u_0$ and $v_0$.

Since this problem has not yet been fully treated, we study its approximation here, in which we replace condition (2) on $S$ by the nonlinear boundary conditions

$$|\nabla u|^{p-1}\nabla u\cdot(1,0) + L\left(Mu^\omega - v\right) = 0, \quad -|\nabla v|^{r-1}\nabla v\cdot(1,0) + L\left(v - Mu^\omega\right) = 0 \tag{6}$$

for positive $L$. This condition preserves $(2)_1$, however, we are able to satisfy $(2)_2$, in a weak sense, by sending $L \to \infty$, only in the case when $p = r = 1$. We shall refer to (6) as $L$-approximation of (2). Problem (1)-(6) is analyzed in [1].

## Transformation and $L$-approximation

Note that plugging

$$U = |u|^m \operatorname{sign} u \quad \text{and} \quad V = |v|^\sigma \operatorname{sign} v$$

into (1), we see that $U, V$ must satisfy

$$\left.\begin{aligned} \partial_t U - \nabla\cdot\left(\vartheta^-(U,\nabla U)\nabla U\right) &= 0 \quad \text{in } Q^-\,, \\ \partial_t V - \nabla\cdot\left(\vartheta^+(V,\nabla V)\nabla V\right) &= 0 \quad \text{in } Q^+\,, \end{aligned}\right\}$$

where

$$\vartheta^-(U,\nabla U) = m^{-p}\,|U|^{\frac{(1-m)p}{m}}\,|\nabla U|^{p-1} \quad \text{and} \quad \vartheta^+(V,\nabla V) = \sigma^{-r}\,|V|^{\frac{(1-\sigma)r}{\sigma}}\,|\nabla V|^{r-1}$$

and the $L$-approximation of (2) on $\Gamma$ is

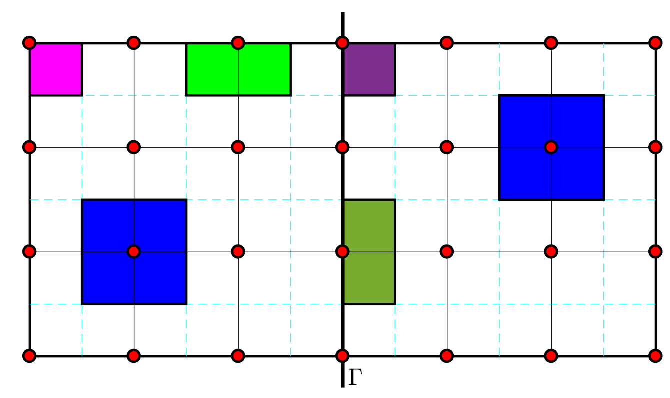
$$\vartheta^-(U,\nabla U)\partial_{x_1}U + L(M^\sigma U^{\frac{\omega\sigma}{m}} - V) = 0, \quad -\vartheta^+(V,\nabla V)\partial_{x_1}V + L(V - M^\sigma U^{\frac{\omega\sigma}{m}}) = 0.$$

## Fully implicit FV scheme

**Space:** finite-volume method with rectangular control volumes using grid points $800\times 800$ on both $\Omega^-$ and $\Omega^+$.

**Time:** backward Euler method, fully implicit in $U, V$ solved by Newton method with analytic Jacobian and damping using residual-based stopping and adaptive $\Delta t$ driven by Newton iteration count.

Numerical experiments are presented mainly in [2].



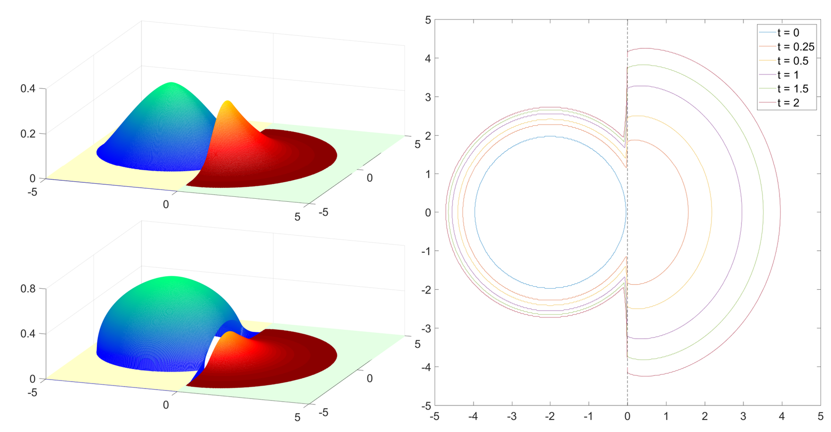

## Numerical experiments

In all experiments we use interface penalty parameter $L = 2\times 10^4$. In figures the upper left panels show $(u,v)$, the lower left show $(U,V)$, and the right panels show time evolution of interface.

**Experiment 1: implicit vs. explicit scheme,** $m = 0.35,\ p = 1.7,\ \sigma = 1.1,\ r = 1.5,\ M = 2.2$ $\omega \approx 0.656$, final time $T = 2$.

**Initial data:** Barenblatt profile on $\Omega^-$ with total mass $R = 10$ and with peak $2$ for $x = (-2,0)$. Right part $\Omega^+$: value induced through the penalized interface.

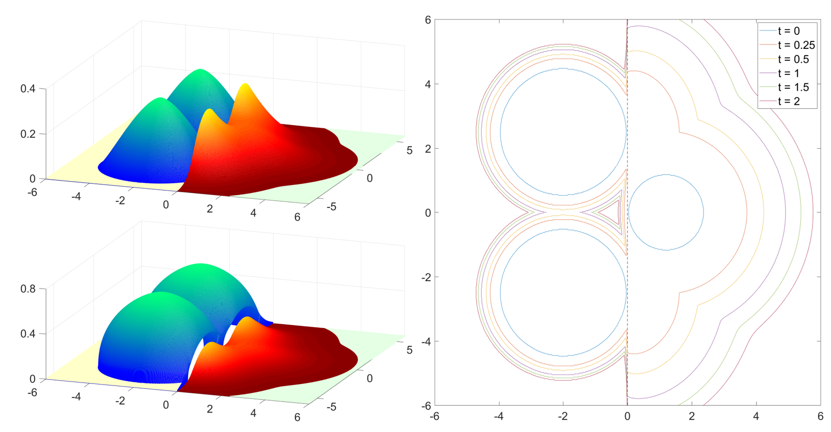
**Main observation:** speedup $\approx 2\times$ vs. explicit scheme with mass loss $< 0.12\%$ and $L^2$–difference of outputs $< 4\times 10^{-6}$.



**Experiment 2: multi-peak left vs. single right,** $m = 0.35,\ p = 1.7,\ \sigma = 1.1,\ r = 1.5,\ M = 2.2$ $\omega \approx 0.656$, final time $T = 2$.

**Initial data:** two equal Barenblatt profiles on $\Omega^-$, each with total mass $R = 10$ and with peak $2$ for $x = (-2, 2.5)$ and $x = (-2,-2.5)$, one Barenblatt profile with peak $1.5$ for $x = (1.2, 0)$ and mass $R = 1$.

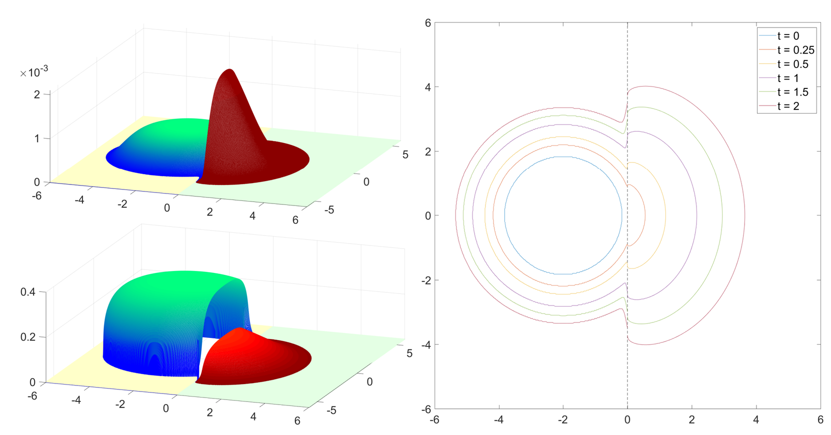
**Main observation:** speedup $\approx 2\times$ vs. explicit scheme, mass loss < 0.13% and comparable differences as before.



**Experiment 3: stiff regime,** $m = 0.15,\ p = 0.3,\ \sigma = 0.35,\ r = 0.4,\ M = 0.5$ $\omega \approx 0.759$, final time $T = 2$.

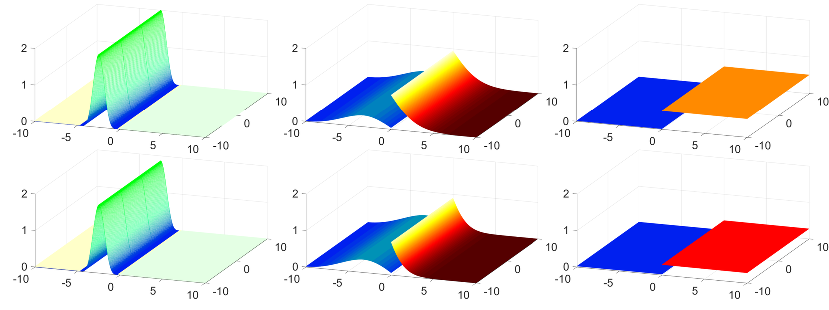
**Initial data:** Barenblatt profile on $\Omega^-$ with total mass $R = 10$ and with peak $2$ for $x = (-2,0)$. Right part again filled only through the interface. Very small exponents $(m,p,\sigma,r) \Rightarrow$ very steep fronts - almost impossible to compute with explicit scheme.

**Main observation:** compute time: $28$ days $18$ hours & mass loss $< 0.17\%$.



**Experiment 4: long-time relaxation,** $m = 0.9,\ p = 1.1,\ \sigma = 1.9,\ r = 2,\ M = 5$ final time $T = 20000$, with Neumann boundary conditions on vertical sides.

**Initial data:** 1D Barenblatt on $\Omega^-$ in $x_1$, constant in $x_2$, peak $2$ for $x_1 = -2.5$ and total mass $R = 60$. We check usefullness of adaptive time step and observe long-time stability of the interface jump.



**Main observation:** at $T = 20000$: approached almost to constants in both subdomains $u \approx 0.01346$, $v \approx 0.5138$, with mass loss $< 0.09\%$.

## References

[1] J. Babušíková, J. Filo, and P. Mihala.
Double nonlinear diffusion equations in a two-component domain.
*Journal of Elliptic and Parabolic Equations, 2025.*

[2] J. Babušíková, J. Filo, and P. Mihala.
Fully implicit finite-volume method for doubly-nonlinear diffusion in two-component domain.
*(in preparation).*

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The presented research was supported by the VEGA Grant No. 1/0709/24.
The principal investigator of the grant: Hana Šmitala Mizerová

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