Existence of periodic solutions of a functional differential equation

Poster For MatFyz CONNECTIONS

Introduction

We analyze positive 1-periodic solutions of

$$x'(t) = f(x(t)) + x(\lfloor t \rfloor)g(t - \lfloor t \rfloor), \quad t \geq 0 \quad (1)$$

where $\lfloor . \rfloor : [0, \infty) \to \mathbb{N} \cup \{0\}$ is the standard floor function, i.e. $\lfloor t \rfloor = n$ if $t \in [n, n+1)$ for some $n \in \mathbb{N} \cup \{0\}$.
The problem (1) represents a numeric model that describes temperature changes in a space bounded by outside environment. A solution of problem represents temperature of the space cooled by colder environment; the negative function f represents cooling up to zero. The space is simultaneously warmed by a heating agent inside; the positive function g represents heating. Moreover, the function g is multiplied by the temperature values at discrete points so that the actual temperature is affected by previous temperature.
In our work, we use the comparison principle for differential equations to prove the existence, uniqueness and asymptotic stability of positive 1-periodic solutions.

Preliminaries

The functions f, g satisfy the following conditions:

1. $f : [0, \infty) \to [-\infty, 0)$ is continuously differentiable, $f(0) = 0$ and $f(x) < 0$ for $x > 0$,
2. $g : [0, 1] \to (0, \infty)$ is continuous.

A continuous function $x : [0, \infty) \to \mathbb{R}$ is a solution of the equation (1) if for every $n \in \mathbb{N} \cup \{0\}$, the function x is continuously differentiable over $(n, n+1)$, and the equation (1) is satisfied for $t \in (n, n+1)$.

In order to study $1-$periodic solutions of (1), we will consider the problem

$$x'(t) = f(x(t)) + x(0)g(t), \quad t \in [0, 1], \quad (2)$$
$$x(0) = x(1), \quad (3)$$

where $x'(0)$ and $x'(1)$ are the corresponding one-sided derivatives.

Results

In the proofs, we use the comparison principle along with the following lemma.

Lemma

Let $f(x) = -\eta x$ where $\eta > 0$. Then there exists a unique $\eta^* = \eta^*(g) > 0$ such that for every positive solution x of (2), the following assertions are true:

1. *If $\eta > \eta^*$, then $x(1) < x(0)$.*
2. *If $\eta < \eta^*$, then $x(1) > x(0)$.*
3. *If $\eta = \eta^*$, then $x(1) = x(0)$.*

The value η^* is the unique root of the equation

$$\int_0^1 e^{\eta t}(g(t) - \eta)\,\mathrm{d}t = 0$$

and represents a boundary between stability and unstability of the zero solution.

Theorem

Let f, g satisfy the conditions 1,2, and let η^* be as in the lemma. Assume that

$$\lim_{x \to 0^+} \frac{f(x)}{x} > -\eta^* > \lim_{x \to \infty} \frac{f(x)}{x}.$$

Then there exists a positive solution x of problem (2)-(3), hence a $1-$periodic solution of (1).
Moreover, if the function $\frac{f(x)}{x}$ is decreasing on interval $(0, \infty)$, then the positive $1-$periodic solution x is unique and for every solution y of (1) such that $y(0) > 0$, it holds

$$\lim_{t \to \infty} (x(t) - y(t)) = 0. \quad (4)$$

References

M. Fečkan, J. Pačuta. Existence and uniqueness of positive periodic solutions of a certain functional differential equation, Electron. J. Qual. Theory Differ. Equ. 2025, No. 62, 1-8.

Example

Consider the equation

$$x'(t) = -\frac{x(t)}{3} - x^2(t) + x(\lfloor t \rfloor)e^{-(t - \lfloor t \rfloor)}, \quad t \geq 0. \quad (5)$$

Due to the results, there is a unique positive 1-periodic solution x.

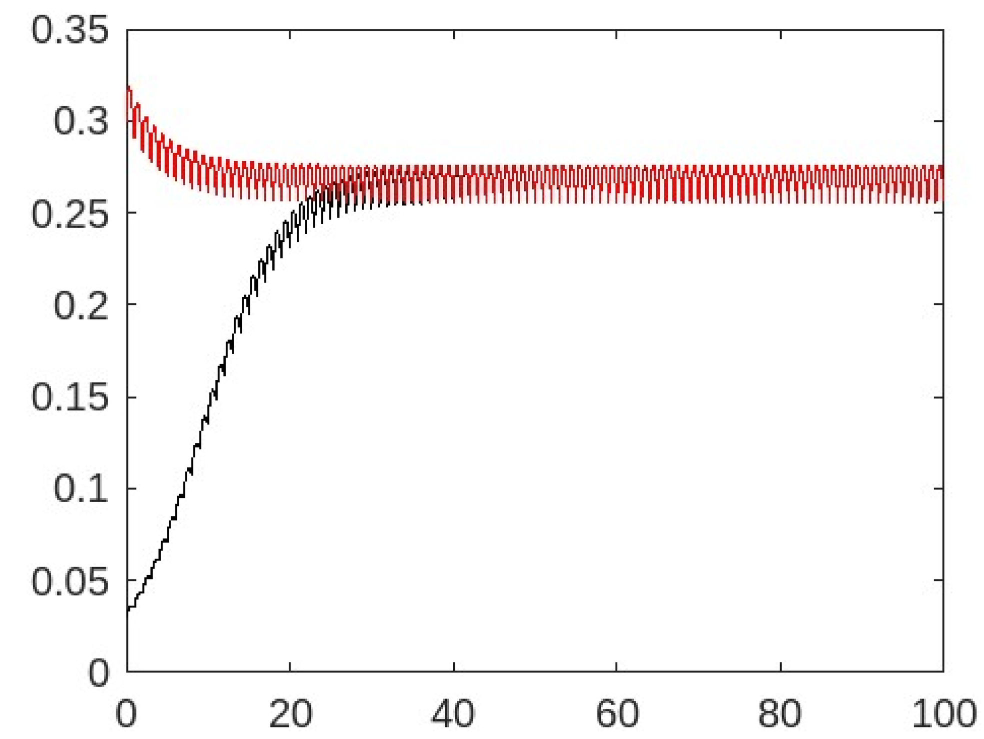

Figure 1. Comparison of solutions x_- (black line) and x_+ (red line).

The equation can be solved numerically and we find that $\eta^* \approx 0.601$. In the Figure 1, we see two solutions $x_{-,+}$ of (5) over the interval $[0, 100]$. For x_-, we chose $x_-(0) = 0.03$ and for x_+, $x_-(0) = 0.3$. The function x_- (x_+) is a subsolution (supersolution) of (2) and $x_-(n)$, $x_+(n)$ for integer n converge to a value that is the initial condition for the periodic solution. Numerical computations show that $x_-(100) \approx 0.256$ and $x_+(100) - x_-(100) \approx 1.64 \times 10^{-9}$. This means that $x_-(100)$ approximates the initial value of the 1-periodic solution with error 1.64×10^{-9}.

The problem (5) describes temperature changes in a space bounded by outside cold environment for particular functions f and g. We see that for such f, g, the optimal initial temperature is approximately equal to 0.256 and repeats regularly at every positive integer point. If the initial temperature is higher or lower, then, after some time, it becomes close to optimal due to the asymptotic stability of the periodic solution in sense of (4).

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