Total colouring of (sub)cubic Halin graphs.

Halin graphs

A graph H is **Halin** if it can be obtained from a plane embedding of a tree T with at least three leaves by adding a cycle passing through all the leaves with no crossing edges. Every Halin graph is connected and planar.

A vertex corresponding to a leaf in T is called **peripheral** or **ring** vertex in H. A vertex which is not peripheral is **spanning**.

An edge connecting two peripheral vertices in H is called **peripheral**. An edge which is not peripheral is called **spanning**.

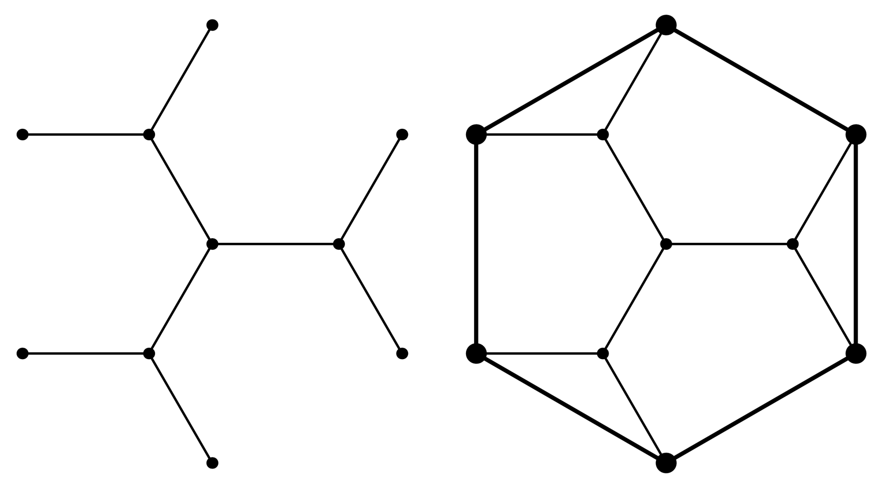

Figure 1. A cubic Halin graph (right) together with the tree it originated from (left).

Total colouring

Let $c'' : V(G) \cup E(G) \to M$ be a mapping from the set of vertices and edges to the set of colours M such that no two incident or adjacent objects receive the same colour. Then c'' is said to be a total colouring of G. The smallest $k = |M|$ for which G has a total colouring is called the total chromatic index $\chi''(G)$.

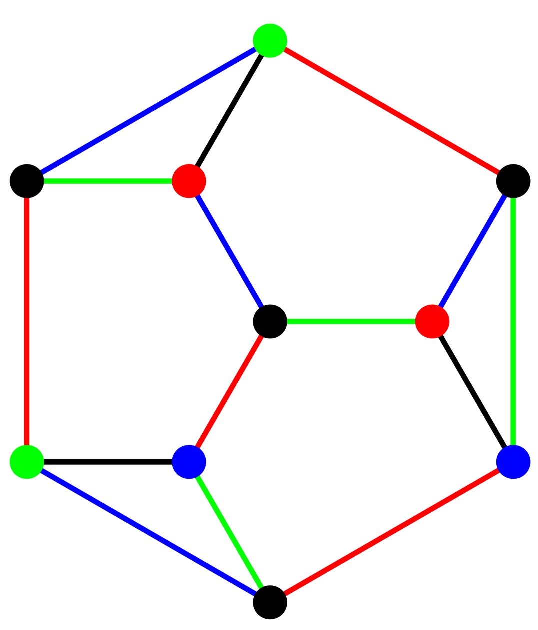

Figure 2. Example of a totally 4-coloured graph.

Halin graphs with $\chi''(G) = 5$

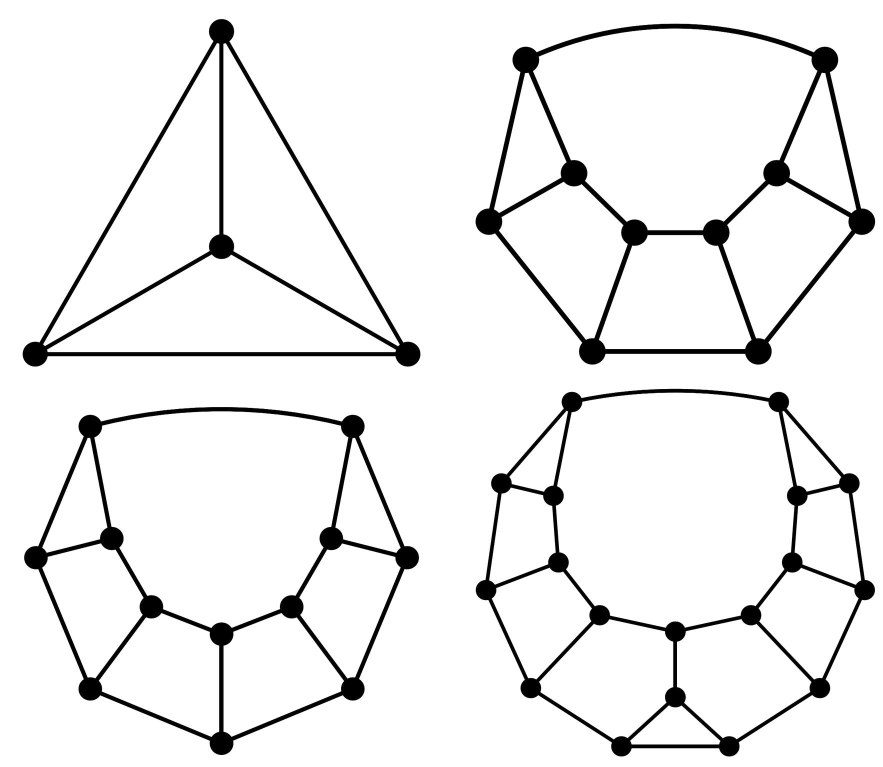

Figure 3. Set of cubic Halin graphs with $\chi''(G) = 5$, containing graphs K_4 (top-left), H_4 (top-right), H_5 (bottom-left) and H_8 (bottom-right).

Motivation

Question: What is the number of cubic Halin graphs with $\chi'' = 5$?

Halin tripoles

Let X, Y be a partition of the vertex set of a cubic Halin graph H such that $E(X, Y)$ contains precisely 1 spanning and 2 peripheral edges. Then X (or Y) is a **Halin tripole**.

If $Y = \{v\}$ contains exactly one peripheral vertex of H, then the tripole X can be denoted as T_v^H.

The **rank** $r(T)$ of a tripole T is the number of spanning vertices in T.

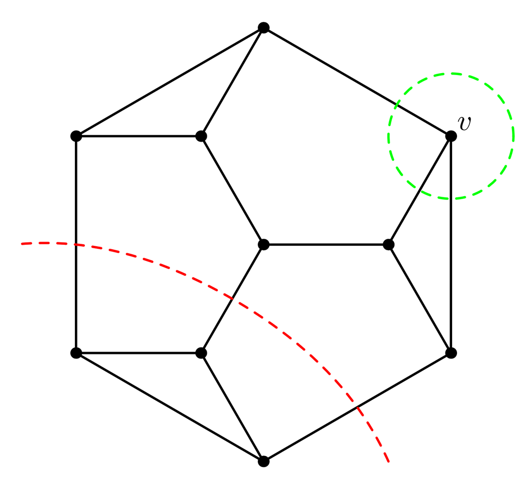

Figure 4. Examples of 3-edge cuts yielding Halin tripoles.

Palettes

Let T be a cubic Halin tripole and let $\{r, p_1, p_2\}$ be its semi-edges and c'' be a total 4-colouring of T. Then the sextuple

$$(c''(r), c''(r^*), c''(p_1), c''(p_1^*), c''(p_2), c''(p_2^*),)$$

is said to be an **extendable colouring** of T.

The complete set of extendable colourings of a tripole T, is said to be is its palette, denoted as $\mathcal{P}(T)$.

Composition and completability

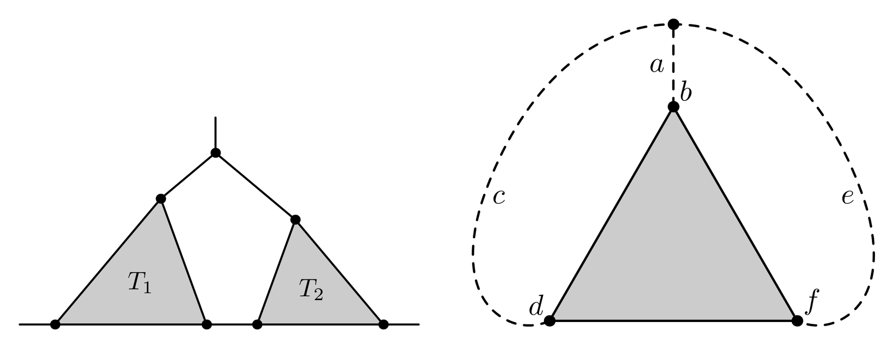

Figure 5. Composition of two Halin tripoles (left) and colouring completability requirements (right).

Results

Theorem 1: The decision problem of total 4-colouring of cubic Halin graphs has an algorithm with linear-time complexity.

Lemma 1: The set of all realizable palettes of cubic Halin tripoles and subcubic Halin tripoles is of size 1213 and 3195 respectively.

Theorem 2: Let H be a cubic Halin graph other than K_4, H_4, H_5 and H_8. Then $\chi''(H) = 4$.

Theorem 3: Let H be a subcubic Halin graph other than $K_4, H_{2,1}, H_4, H_5$ and H_8. Then $\chi''(H) = 4$.

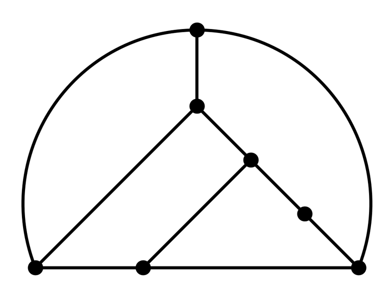

Figure 6. Graph $H_{1,2}$

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