Circular chromatic index

Computational methods for Upper Gap Conjecture

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Motivation: Circular colourings model scheduling optimization problems where we don't require fixed time slots (unlike colourings with integer-valued colours).

A **circular r-edge-colouring** c assigns colours from $[0, r)$ to edges in such a way that for any two adjacent edges e and f we have $1 \leq |c(e) - c(f)| \leq r - 1$.

Circular chromatic index (investigated since about 2000):

$$\chi'_c(G) = \inf \{r \mid G \text{ has circular } r\text{-edge-colouring}\}.$$

- Infimum always attained and rational; $\chi'(G) = \lceil \chi'_c(G) \rceil$.
- There is a finite set of fractions p/q to consider for a given graph ($p \leq |E(G)|$).
- According to the Vizing theorem, chromatic index $\chi'(G)$ of a Δ-regular graph G is Δ or $\Delta + 1$, hence $\chi'_c(G) \in [\Delta, \Delta + 1]$.
- In a Δ-edge-colouring of G, colours of edges around a vertex are given; we can only change the ordering of the colours. In a $(\Delta + \varepsilon)$-edge-colouring, that extra ε allows the colours to shift slightly around each vertex (they no longer have to be integers). Over a large graph, these small shifts accumulate, and typically a small ε is sufficient.
- Graphs of large enough girth are *all* $(\Delta + \varepsilon)$-colourable for any small ε [1, 2].
- It is easy to construct sequences of values of χ'_c decreasing towards Δ. It was conjectured by Zhu [3] that there are no increasing sequences; we managed to refute that in 2010 [4].

What values can χ'_c take?

- For $\Delta = 3$, every rational in $[3, 3 + 1/3]$ is attainable [4].
- Infinitely many values in $[3 + 1/3, 3 + 1/2]$ (the only accumulation point is $3 + 1/3$), no graphs known for $(3 + 1/2, 3 + 2/3)$ (open). The middle portion of the interval $[\Delta, \Delta + 1]$ seems to be the hardest.
- The Petersen graph has $\chi'_c = 3 + 2/3$, no other such graph known (open).
- No graphs or multigraphs with $\chi'_c \in (3 + 2/3, 4)$ exist [5].
- For $\Delta \geq 4$, every rational in $[\Delta, \Delta + 1/6]$ is attainable (can be extended to $\Delta + 1/3$ if multigraphs are allowed) [6].
- Infinitely many additional values below $\Delta + 1/2$ [7], but nothing exhaustive or systematic known.

Upper Gap Conjecture (UGC)

Conjecture: For any Δ, there are no graphs with $\chi'_c(G) \in (\Delta + 1 - 1/\Delta, \Delta + 1)$.

Status: Proved for $\Delta \in \{2, 3\}$. Open for $\Delta \geq 4$.

We conducted a systematic computation of circular chromatic index of all small graphs for $\Delta \in \{4, 5, 6\}$ with two aims:
(1) counterexamples to Upper Gap Conjecture;
(2) graphs with $\chi'_c = \Delta + 1$, because these graphs are obstacles to any statement of the form "all graphs are $(\Delta + \varepsilon)$-edge-colourable for some $\varepsilon < 1$" (which is the crux of UGC) and have to be excluded by some extra assumption (there are only two such graphs for $\Delta = 3$, so this extra assumption is trivial in that case).

For larger orders, we discarded graphs that were $(\Delta + 1/2)$-edge-colourable to save on computation time, so we likely missed some interesting fractions.

Computational challenges

- Determining χ' and χ'_c is NP-hard. So is subgraph isomorphism.
- Computing χ'_c takes much longer than computing χ' because the number of colours grows with graph size even for fixed Δ.
- There are existing generators for graphs and multigraphs with maximum degree Δ, but none gives any control over colourability. We mostly used the CVD heuristic to filter out colourable graphs.
- Generally, the fastest approach is reduction to SAT (and using Kissat as the solver, occasionally with breakid for symmetry breaking). Sometimes it is highly advantageous to run the VF2 algorithm first, filtering out graphs that contain specific small subgraphs enforcing large χ'_c. MILP is by far worse, but CP-SAT from Google seems faster than SAT specifically for circular $9/2$-edge-colouring. No single method dominates in every situation.
- The critical parameter for SAT is the number of variables, which grows quadratically with graph size in our case (not good). However, Δ strongly affects running time: for $\Delta = 3$, a graph with $\sim$50 edges takes minutes, for $\Delta \geq 6$ even 30 edges might take weeks.
- UGC is computationally infeasible for $\Delta \geq 6$. The number of graphs grows about 100x per vertex and the computation of p/q-colouring even with $q = 3$ takes days for graphs with 10 vertices.

Results: $\Delta = 4$

Order	Multigraphs	New values of χ'_c
3	5	$5/1$, $6/1$
4	25	—
5	124	$9/2$
6	704	—
7	4283	$13/3$, $14/3$
8	29773	—
9	227016	$17/4$
10	1916000	—
11	17633748	$21/5$, $22/5$, $19/4$
12	176094228	—
13	1893482910	$23/5$

Order	Graphs	New values of χ'_c
5	11	$9/2$, $5/1$
6	49	—
7	289	—
8	1735	—
9	11676	$13/3$, $14/3$
10	87669	—
11	733811	—
12	6781207	—
13	68462296	$17/4$, $22/5$
14	748330892	—
15	8787966433	— (above $9/2$)

We found two 4-connected 4-regular graphs with $\chi'_c = 5$ and constructed infinite families of graphs with $\chi'_c = \Delta + 1$ for $\Delta \in \{4, 5, 6\}$, refuting all UGC variants using extra edge-connectivity assumption suggested in [8].

In stark contrast to $\Delta = 3$, (multi)graphs with $\chi'_c \in \{\Delta + 2/3, \Delta + 1\}$ seem plentiful for $\Delta \geq 4$. The larger the degree, the more so.

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MATFYZ CONNECTIONS

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