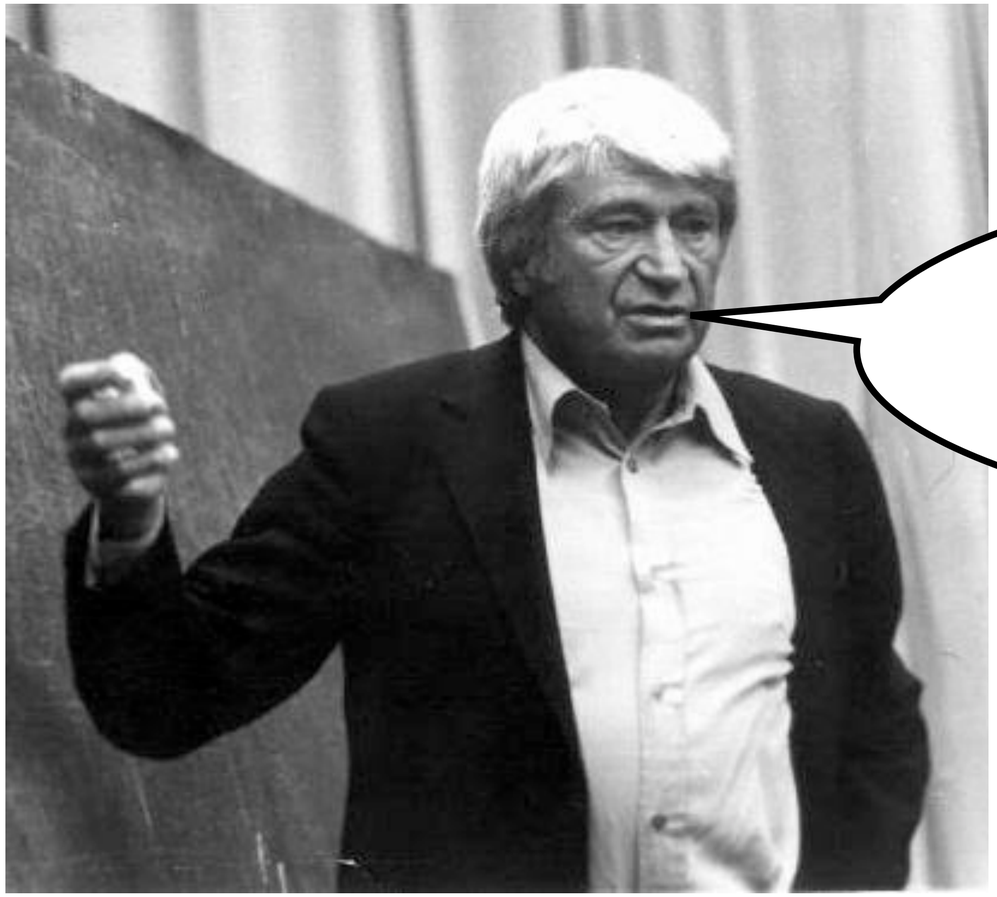
Discrete 2026

Adiabatic Response in the Migdal Effect

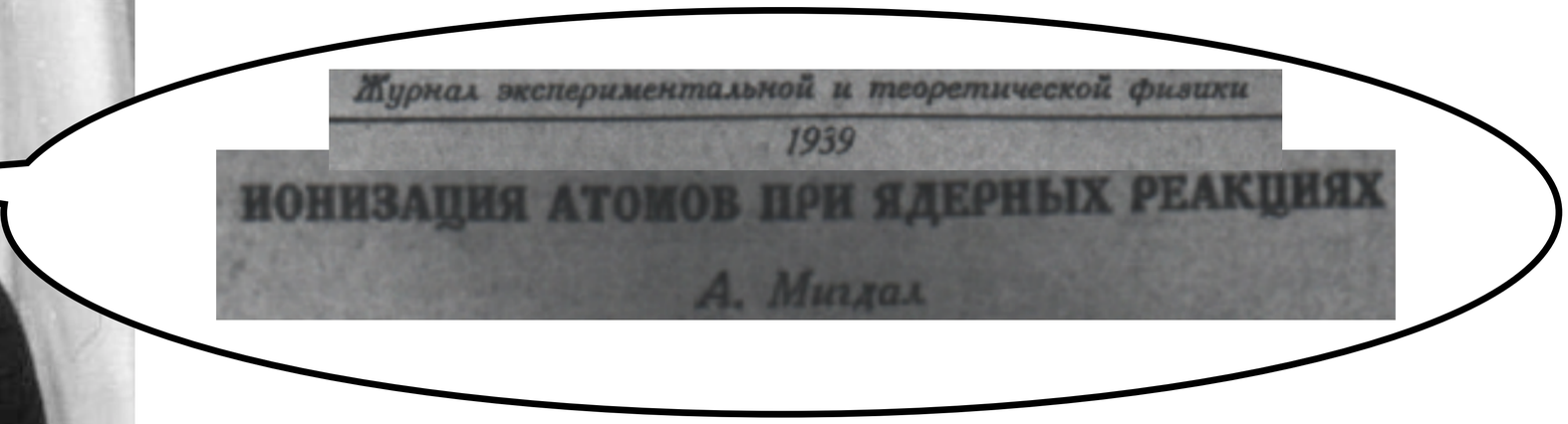
Stefan Daniel Nellen Mondragón in collaboration with *Josef Pradler and Mukul Sholapurkar*
Particle Physics Group, University of Wien & MBI-ÖAW
14th of November 2026

Based on *Adiabatic response in the Migdal Effect*; 2606.14845v1

What is the Migdal effect?



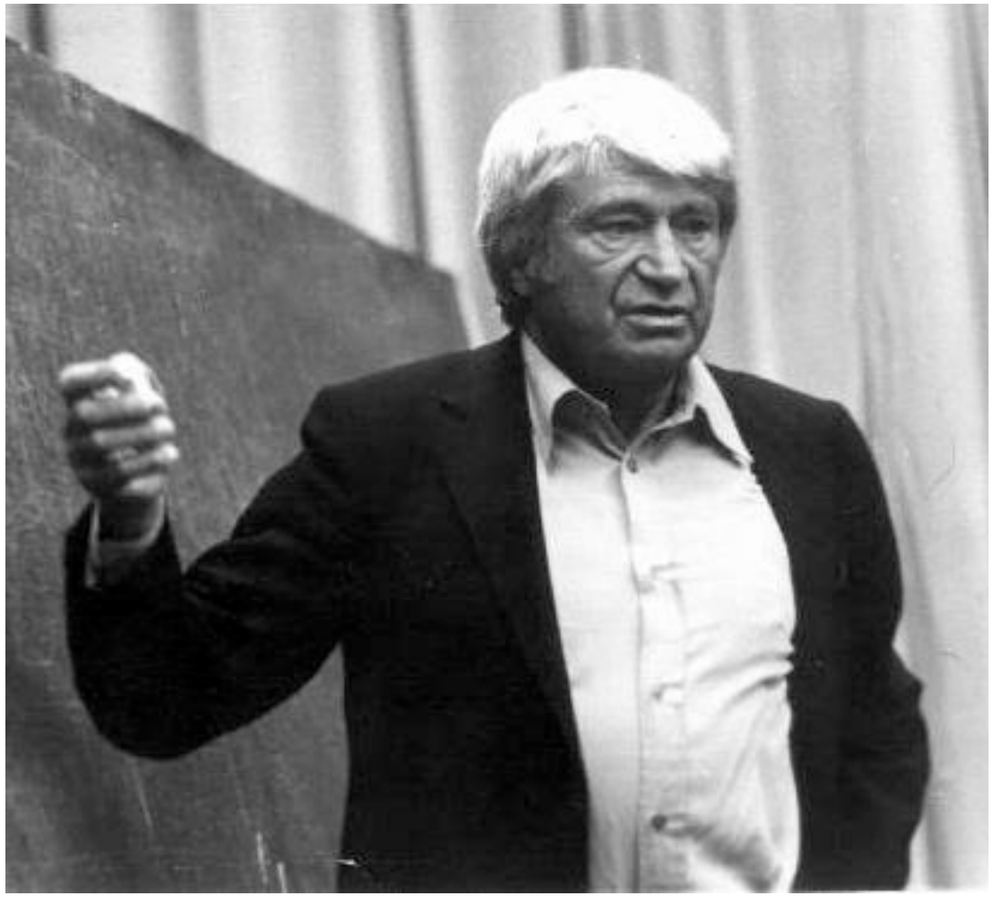
Arkady Migdal



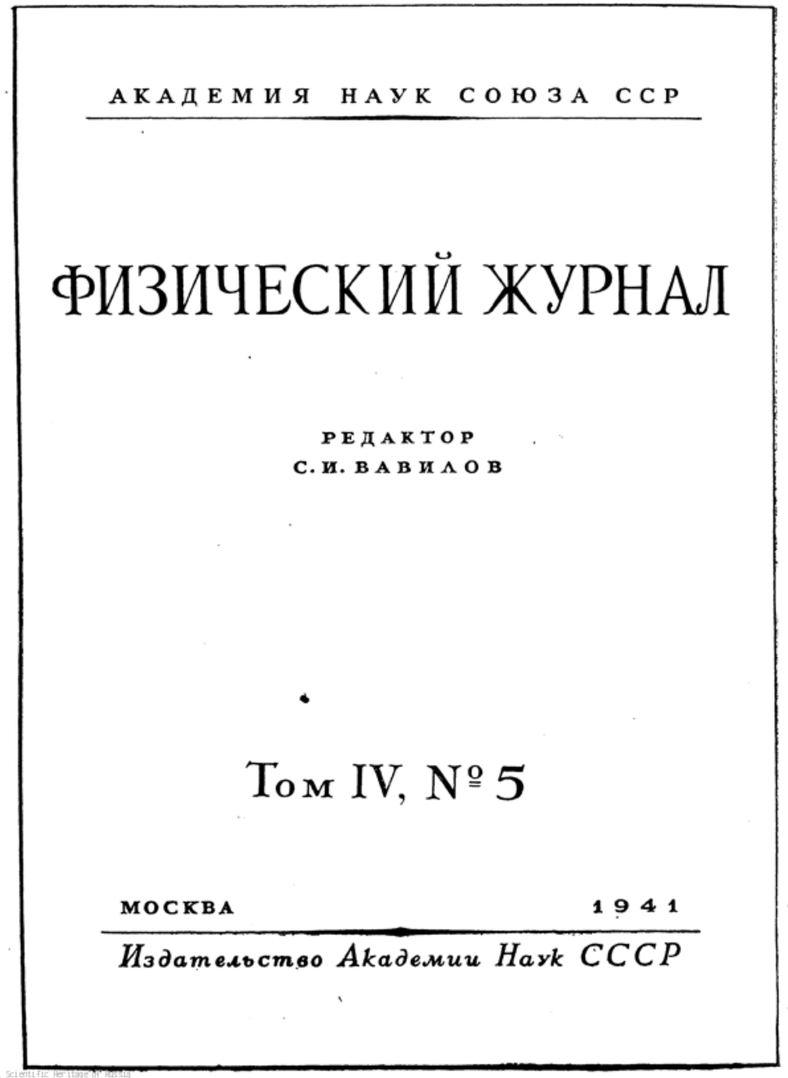
Ionization of atoms in nuclear reactions

A. Migdal. *Ionizatsiya atomov pri yadernykh reaktsiyakh*. J. Phys. Acad. Sci. USSR 9, 423-432, 1163–1165 (1939)

What is the Migdal effect?



Arkady Migdal



АКАДЕМИЯ НАУК СОЮЗА ССР

ФИЗИЧЕСКИЙ ЖУРНАЛ

РЕДАКТОР
С. И. ВАВИЛОВ

Том IV, № 5

МОСКВА 1941

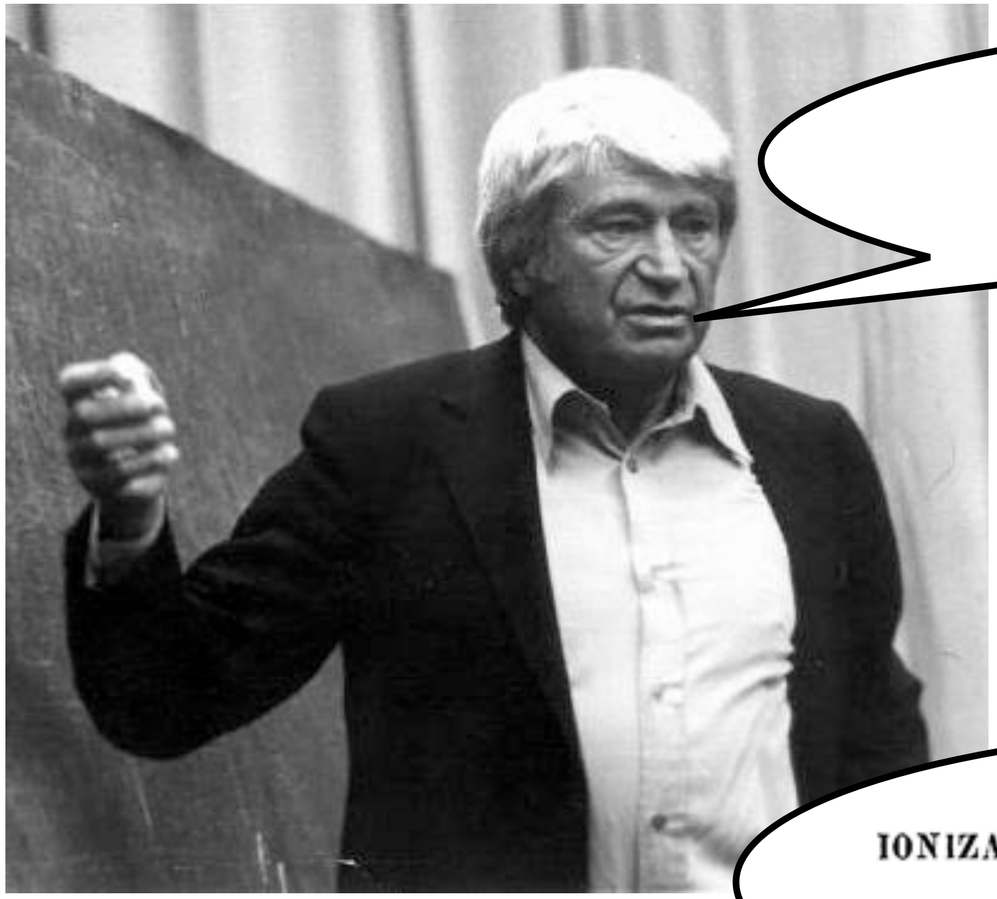
Издательство Академии Наук СССР



Evgenii Feinberg

J. Phys. Acad. Sci. USSR 4, 423-432, 449–453 (1941)

What is the Migdal effect?



Arkady Migdal

IONIZATION OF ATOMS ACCOMPANYING α- and β-DECAY

By A. MIGDAL

(Received November 15, 1940)

IONIZATION OF THE ATOM DUE TO β-DECAY*

By E. L. FEINBERG

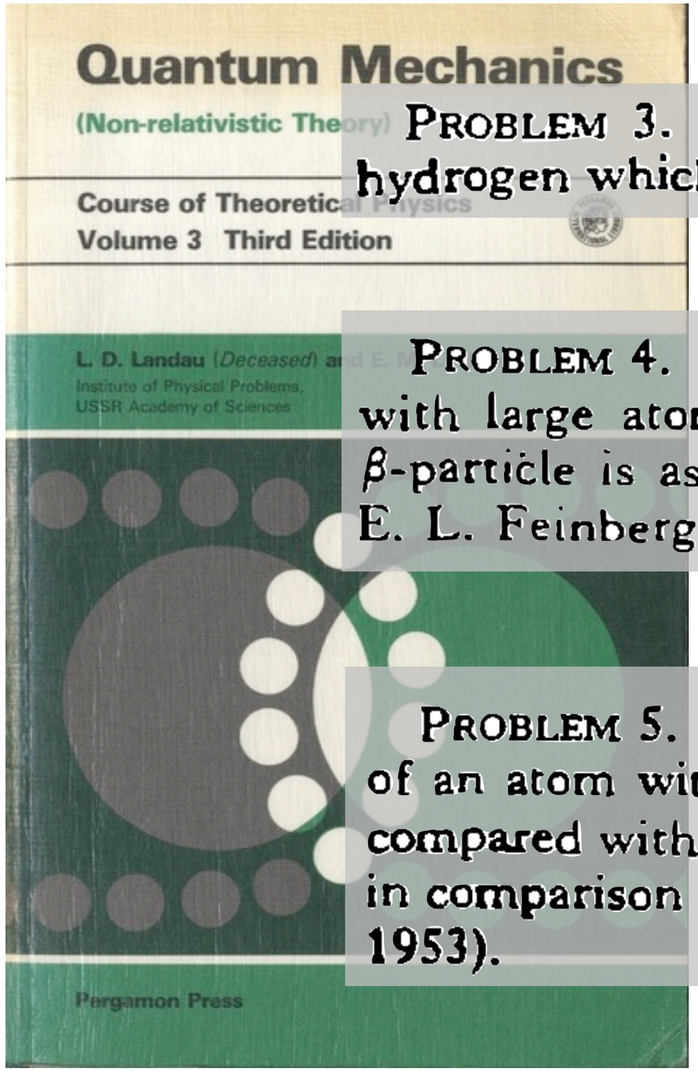
(Received September 26, 1940)



Evgenii Feinberg

J. Phys. Acad. Sci. USSR 4, 423-432, 449–453 (1941)

The Migdal effect - your homework



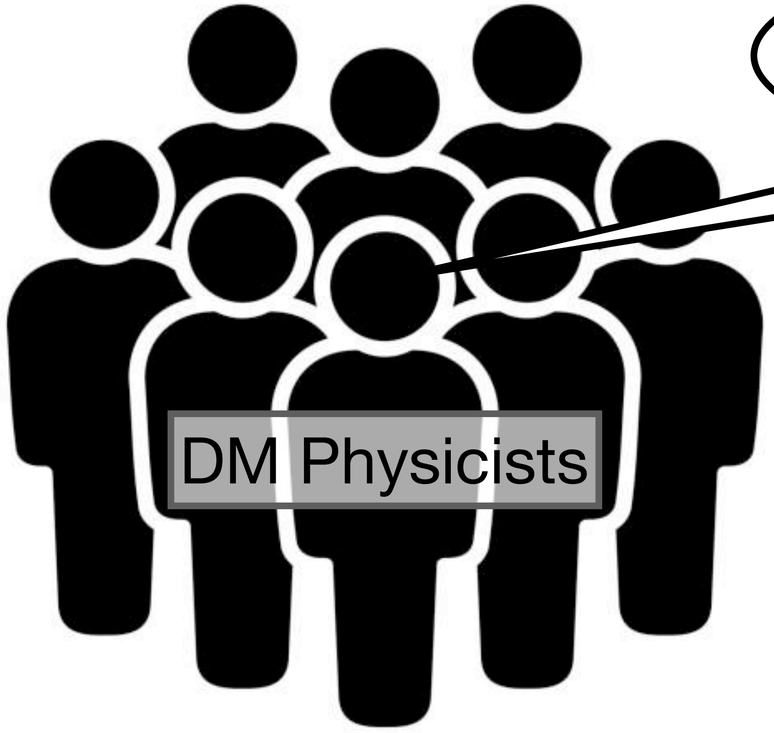
PROBLEM 3. Determine the total probability of excitation and ionization of an atom of hydrogen which receives a sudden "jolt" (see Problem 2).

PROBLEM 4. Determine the probability that an electron will leave the K-shell of an atom with large atomic number Z when the nucleus undergoes β-decay. The velocity of the β-particle is assumed large in comparison with that of the K-electron (A. B. Migdal and E. L. Feinberg 1941).

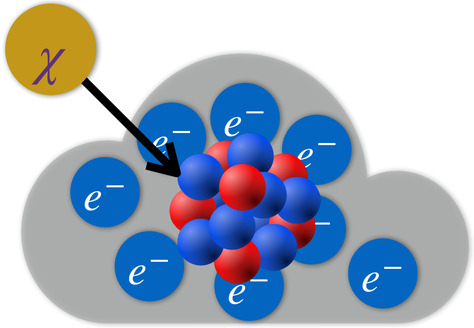
PROBLEM 5. Determine the probability of emergence of an electron from the K-shell of an atom with large Z in α-decay of the nucleus. The velocity of the α-particle is small compared with that of the K-electron, but the time which it takes to leave the nucleus is small in comparison with the time of revolution of the electron (A. B. Migdal 1941, J. S. Levinger 1953).

L. Landau and E. Lifshitz. *Quantum Mechanics : Non-relativistic Theory.* Pergamon Press, Oxford, 3rd Ed. (1958, 1977)

What do *we* mean by the Migdal effect?

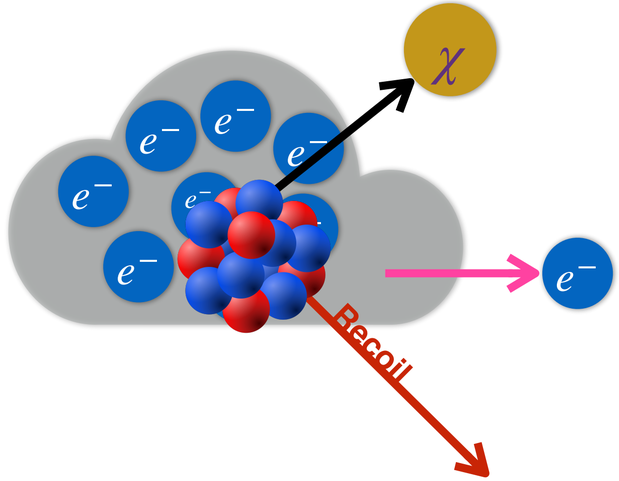


*The prompt ionization of an atom**
following the scattering/recoil of its nucleus



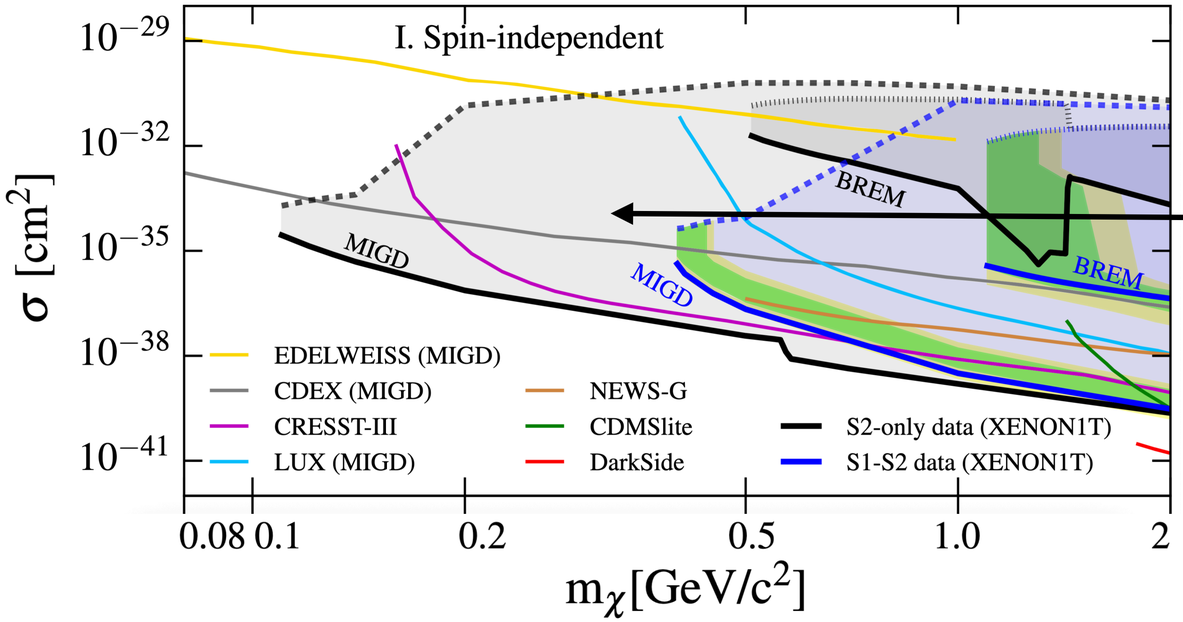
Atom recoils &

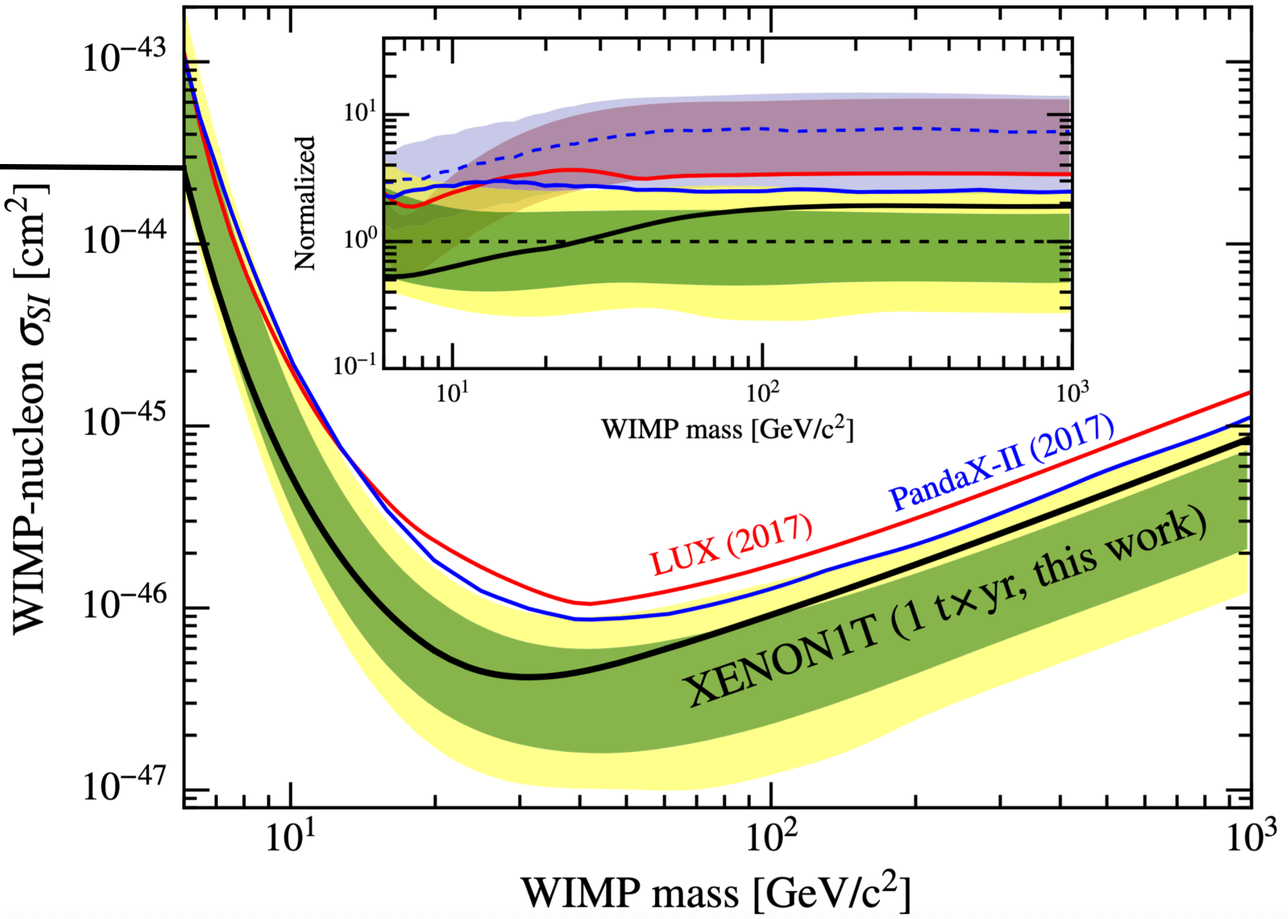
ionizes



**or something more general, eg. the electronic response of a semiconductor*

DM direct detection - why do we care about Migdal?





Inelastic channel extends reach of detector by more than an order of magnitude — and for free!

Competitive with dedicated low mass detectors

Figures from:
E. Aprile et al. (XENON), *Dark Matter Search Results from a One Ton-Year Exposure of XENON1T*, Phys. Rev. Lett. 121, 111302 (2018)
E. Aprile et al. (XENON), *Search for Light Dark Matter Interactions Enhanced by the Migdal Effect or Bremsstrahlung in XENON1T*, Phys. Rev. Lett. **123** (2019)

Calculating the Migdal rate - the impulse approximation

The electrons *can not* adjust to the recoiling nucleus during scattering

$$\langle \phi_f | \exp\left(-im_e \sum_{j=1}^{Z} \mathbf{v}_f \cdot \hat{\mathbf{x}}_j \right) | \phi_i \rangle$$

Instantaneously: **Electronic transition operator is a Galilean boost**

$$\langle \phi_f | \exp\left(i\beta \sum_{j=1}^{Z} \mathbf{q} \cdot \hat{\mathbf{x}}_j \right) | \phi_i \rangle$$

$\frac{m_e}{m_N}$

DM-nucleus momentum transfer

Dipole approximation: $\beta q r_A \ll 1$

$$i\frac{\beta}{e}\mathbf{q} \cdot \langle \phi_f | e \sum_{j=1}^{Z} \hat{\mathbf{x}}_j | \phi_i \rangle$$

$$\frac{\mathrm{d}\sigma_{n\ell \to k}}{\mathrm{d}E_R \mathrm{d}\omega} = \frac{\mathrm{d}\sigma_{\chi N}}{\mathrm{d}E_R} \frac{1}{2\pi} \frac{\mathrm{d}p_{n\ell \to k}}{\mathrm{d}\omega}$$

Factorized cross section

The adiabatic theorem

Beweis des Adiabatensatzes.

Von **M. Born** und **V. Fock** in Göttingen.

(Eingegangen am 1. August 1928.)

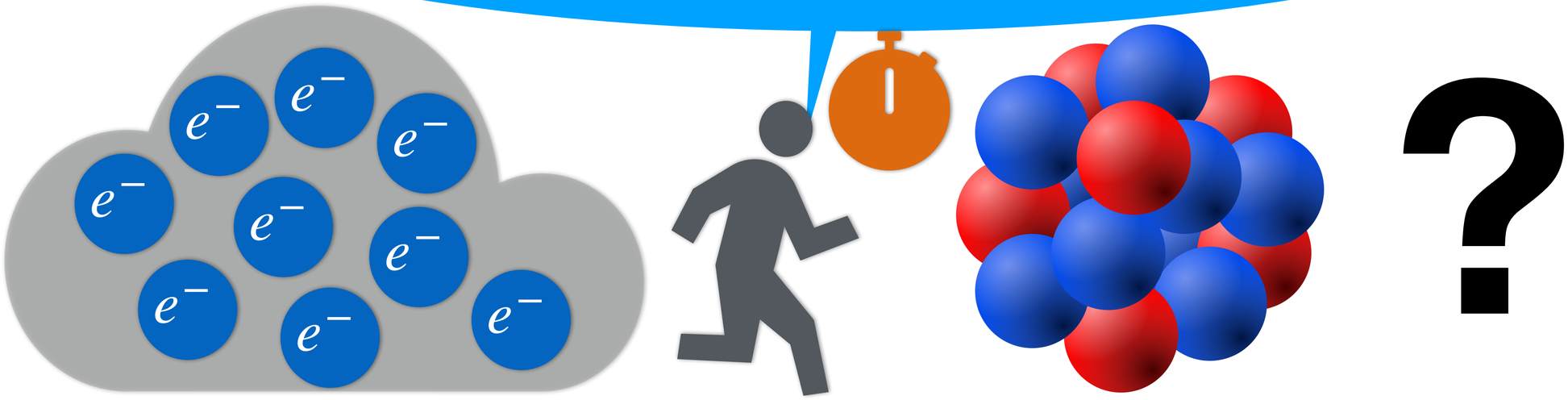
mechanik. Numeriert man die Zustände eines Systems mit den Nummern der entsprechenden Energieniveaus, so behauptet der Adiabatensatz, daß, falls das System sich anfangs in einem Zustand mit einer bestimmten Nummer befand, bei einer adiabatischen Änderung die Wahrscheinlichkeit des Übergangs des Systems in einen Zustand mit einer anderen Nummer unendlich klein ist, trotzdem die Energieniveaus nach der Änderung sich von ihren Anfangswerten um endliche Größen unterscheiden können.

A gapped **eigenstate remains in a** corresponding **eigenstate** after adiabatic evolution.*

M. Born & V. Fock, *Beweis des Adiabatensatzes.* Zeitschrift für Physik A. 51, 165-180 (1928)

The adiabatic theorem and the Migdal effect

How long does it take for the electron cloud to catch up to the recoiling nucleus?



Calculating the Migdal rate - from a Hamiltonian

$$\widehat{H}_0 = \frac{\hat{\mathbf{p}}_N^2}{2m_N} + \frac{\hat{\mathbf{p}}_\chi^2}{2m_\chi} + \sum_{j=1}^{Z} \frac{\hat{\mathbf{p}}_{e,j}^2}{2m_e} + \sum_{j=1}^{Z} V_{eN}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_N) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_{e,k})$$

$$\widehat{H}_e\,|\phi_a\rangle = E_a\,|\phi_a\rangle$$

Introduce perturbing potential $V_{\chi N}(\hat{\mathbf{x}}_\chi - \hat{\mathbf{x}}_N)$ and asymptotic states $|i/f\rangle = |\mathbf{p}_{\chi,\,i/f}\rangle\,|\mathbf{p}_{N,\,i/f}\rangle\,|\phi_{i/f}\rangle$

$$T_{fi} = (2\pi)^3\ \widetilde{V}_{\chi N}(\mathbf{q})\ \delta^3\big(\mathbf{q} + (\mathbf{p}_{N,i} - \mathbf{p}_{N,f})\big)\langle\phi_{a_f}(\{\mathbf{x}'_{e,j}\})\,|\,e^{i\frac{m_e}{m_N}(\mathbf{p}_{N,i}-\mathbf{p}_{N,f})\cdot\sum_j \hat{\mathbf{x}}'_{e,j}}\,|\,\phi_{a_i}(\{\mathbf{x}'_{e,j}\})\rangle$$

The impulse approximation creeps up again

$\mathbf{x}_{e,j} - \mathbf{x}_N$

Going beyond Born - Jacobi coordinates

$$\widehat{H} = \frac{\hat{\mathbf{p}}_N^2}{2m_N} + \frac{\hat{\mathbf{p}}_\chi^2}{2m_\chi} + \sum_{j=1}^{Z} \frac{\hat{\mathbf{p}}_{e,j}^2}{2m_e} + \sum_{j=1}^{Z} V_{eN}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_N) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_{e,k}) + V_{\chi N}(\hat{\mathbf{x}}_\chi - \hat{\mathbf{x}}_N)$$

Doubly relative coordinates

$$\mathbf{R} = \frac{1}{M_{\text{tot}}}\left(m_e \sum_j \mathbf{x}_{e,j} + m_N \mathbf{x}_N + m_\chi \mathbf{x}_\chi\right)$$

$$\mathbf{r}_j = \mathbf{x}_{e,j} - \mathbf{x}_N$$

$$\rho = \mathbf{x}_\chi - \mathbf{R}_{eN}$$

$$\mathbf{p}_R = \mathbf{p}_\chi + \mathbf{p}_N + \sum_j \mathbf{p}_{e,j}$$

$$\mathbf{p}_{r,j} = \mathbf{p}_{e,j} - \frac{m_e}{M_A}\left(\mathbf{p}_N + \sum_k \mathbf{p}_{e,k}\right)$$

$$\mathbf{p}_\rho = \frac{M_A}{M_{\text{tot}}}\mathbf{p}_\chi - \frac{m_\chi}{M_{\text{tot}}}\left(\mathbf{p}_N + \sum_j \mathbf{p}_{e,j}\right)$$

$$\widehat{H} = \frac{\hat{\mathbf{p}}_R^2}{2M_{\text{tot}}} + \frac{\hat{\mathbf{p}}_\rho^2}{2\mu_{\chi A}} + \sum_j \frac{\hat{\mathbf{p}}_{r,j}^2}{2\mu_{eN}} + \sum_j V_{eN}(\hat{\mathbf{r}}_j) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k) + V_{\chi N}(\hat{\mathbf{r}} + \beta\hat{\rho})$$

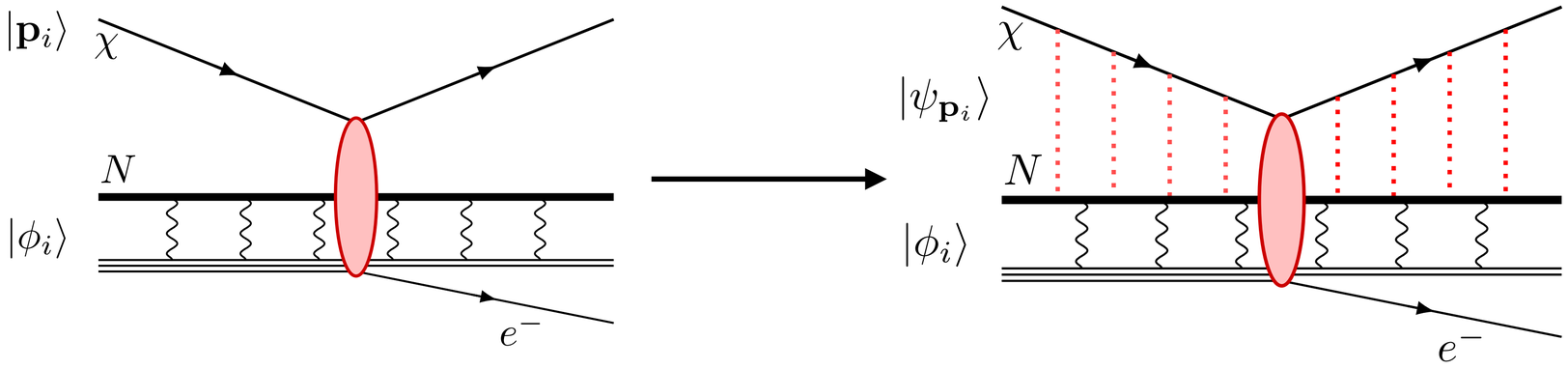
$$\beta = \frac{m_e}{M_A}$$

Going beyond Born - Jacobi coordinates

$$\widehat{H} = \frac{\hat{\mathbf{p}}_R^2}{2M_{\rm tot}} + \frac{\hat{\mathbf{p}}_\rho^2}{2\mu_{\chi A}} + \sum_j \frac{\hat{\mathbf{p}}_{r,j}^2}{2\mu_{eN}} + \sum_j V_{eN}(\hat{\mathbf{r}}_j) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k) + V_{\chi N}(\hat{\rho} + \beta \sum_j \hat{\mathbf{r}}_j)$$

$$\widehat{H} = \frac{\hat{\mathbf{p}}_R^2}{2M_{\rm tot}} + \frac{\hat{\mathbf{p}}_\rho^2}{2\mu_{\chi A}} + \sum_j \frac{\hat{\mathbf{p}}_{r,j}^2}{2\mu_{eN}} + \sum_j V_{eN}(\hat{\mathbf{r}}_j) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k) + V_{\chi N}(\hat{\rho}) + \beta \nabla_\rho V_{\chi N}(\hat{\rho}) \cdot \sum_j \hat{\mathbf{r}}_j$$

We can treat the **atom** and the **scattering at equal footing**



Induces atomic transitions and inelasticity

Going beyond Born - scattering states

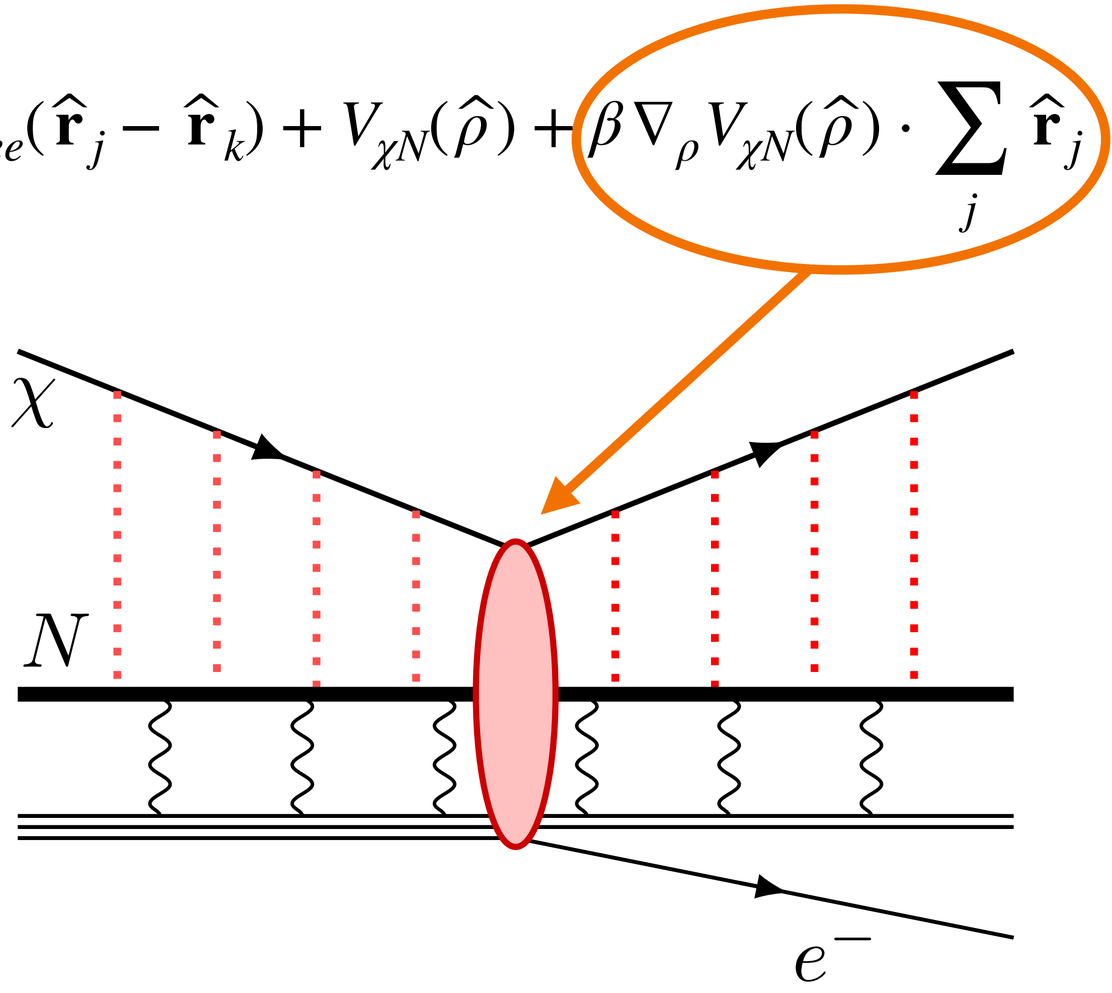
$$\widehat{H} = \frac{\hat{\mathbf{p}}_R^2}{2M_{\text{tot}}} + \frac{\hat{\mathbf{p}}_\rho^2}{2\mu_{\chi A}} + \sum_j \frac{\hat{\mathbf{p}}_{r,j}^2}{2\mu_{eN}} + \sum_j V_{eN}(\hat{\mathbf{r}}_j) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k) + V_{\chi N}(\hat{\rho}) + \beta \nabla_\rho V_{\chi N}(\hat{\rho}) \cdot \sum_j \hat{\mathbf{r}}_j$$

$$|\psi_{\mathbf{p}}^{\pm}\rangle = |\mathbf{p}\rangle + (E_p - \widehat{H}_{\chi N} \pm i0^+)^{-1} \widehat{V}_{\chi N} |\psi_{\mathbf{p}}^{\pm}\rangle \quad |\psi_{\mathbf{p}_i}\rangle$$

Solutions of the Lippmann-Schwinger equation

$$\sum_j \frac{\hat{\mathbf{p}}_{r,j}^2}{2\mu_{eN}} + \sum_j V_{eN}(\hat{\mathbf{r}}_j) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{r}}_j - \hat{\mathbf{r}}_k) \quad |\phi_i\rangle$$

Atomic eigenstates



The Migdal amplitude

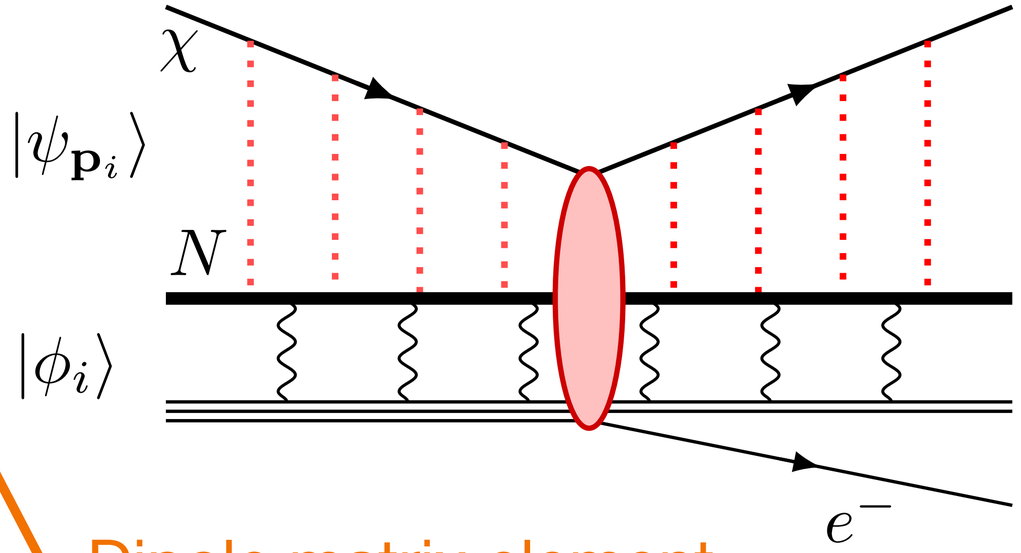
We can compute to first order in the transition operator

Initial and final states $|i/f\rangle = |\psi^{\pm}_{\mathbf{p}_{i/f}}\rangle\,|\phi_{i/f}\rangle$

$$\mathscr{M}_{fi} = \langle f|\,\beta\,\nabla_\rho V_{\chi N}(\hat{\rho})\cdot\sum_j \hat{\mathbf{r}}_j\,|i\rangle = -\frac{\beta}{e}\widetilde{\mathbf{F}}_{fi}(\omega)\cdot\mathbf{d}_{fi}$$

Force matrix element

$$\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i) = -\langle\psi^-_{\mathbf{p}_f}|\,\nabla_\rho V_{\chi N}(\hat{\rho})\,|\psi^+_{\mathbf{p}_i}\rangle$$



Dipole matrix element

$$\mathbf{d}_{fi} = \langle\phi_f|\sum_j e\,\hat{\mathbf{r}}_j\,|\phi_i\rangle$$

The Migdal amplitude - the Born limit

We take the limit of plane wave states

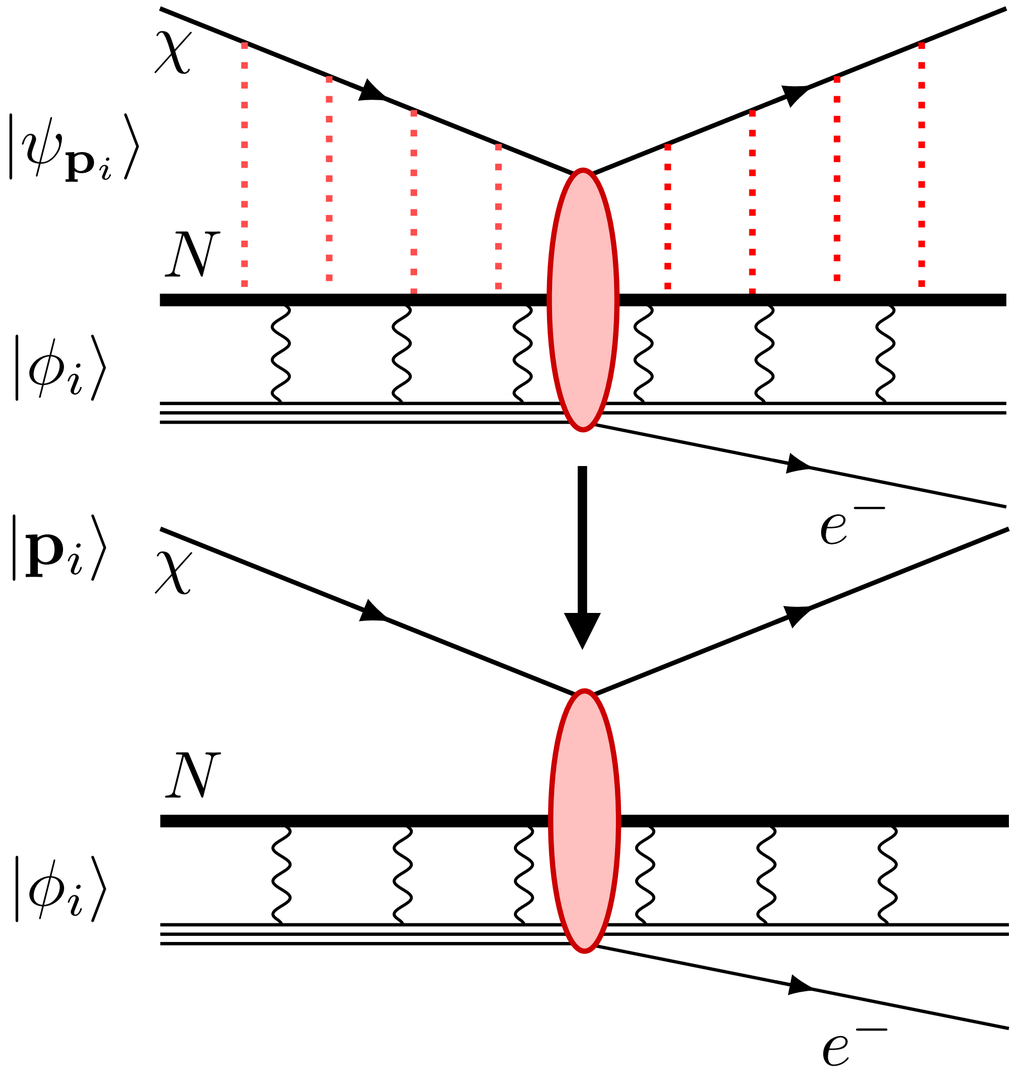
$$|\psi^{\pm}_{\mathbf{p}_{i/f}}\rangle \to |\mathbf{p}_{i/f}\rangle$$

$$\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i) = -\langle \mathbf{p}_f | \nabla_\rho V_{\chi N}(\hat{\rho}) | \mathbf{p}_i \rangle$$

$$\nabla V_{\chi N}(\hat{\rho}) = i[\hat{\mathbf{p}}_\rho, V_{\chi N}(\hat{\rho})]$$

$$\mathscr{M}_{fi} = i\frac{\beta}{e}\widetilde{V}_{\chi N}(\mathbf{q})\,\mathbf{q}\cdot\mathbf{d}_{fi}$$

We recover the impulse approximation!



The differential cross section

$$\mathscr{M}_{fi} = -\frac{\beta}{e}\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i)\cdot\mathbf{d}_{fi}$$

Square, normalize and integrate

For $m_\chi \ll m_N$ Jacobi coordinates → lab coordinates

$$\frac{\mathrm{d}\sigma_{n\ell\to k}}{\mathrm{d}E_R\mathrm{d}\omega} = |F_{\mathrm{ad}}|^2 |F_{\mathrm{el}}|^2 \left.\frac{\mathrm{d}\sigma_{\chi N}}{\mathrm{d}E_R}\right|_{\mathrm{Born}} \frac{1}{2\pi}\frac{\mathrm{d}p_{n\ell\to k}}{\mathrm{d}\omega}$$

Adiabatic form factor

$$|F_{\mathrm{ad}}(\omega/E_i, q/p_i)|^2 = \frac{|\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i)|^2}{q\,|\mathscr{A}_{\mathrm{el}}(q, p_i)|^2}$$

Elastic form factor

$$|F_{\mathrm{el}}(q, p_i)|^2 = \frac{|\mathscr{A}_{\mathrm{el}}(q, p_i)|^2}{|\widetilde{V}_{\chi N}(q)|^2}$$

The adiabatic form factor

$$\mathscr{M}_{fi} = -\frac{\beta}{e}\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i) \cdot \mathbf{d}_{fi}$$

We parameterize the the **magnitude** of amplitude

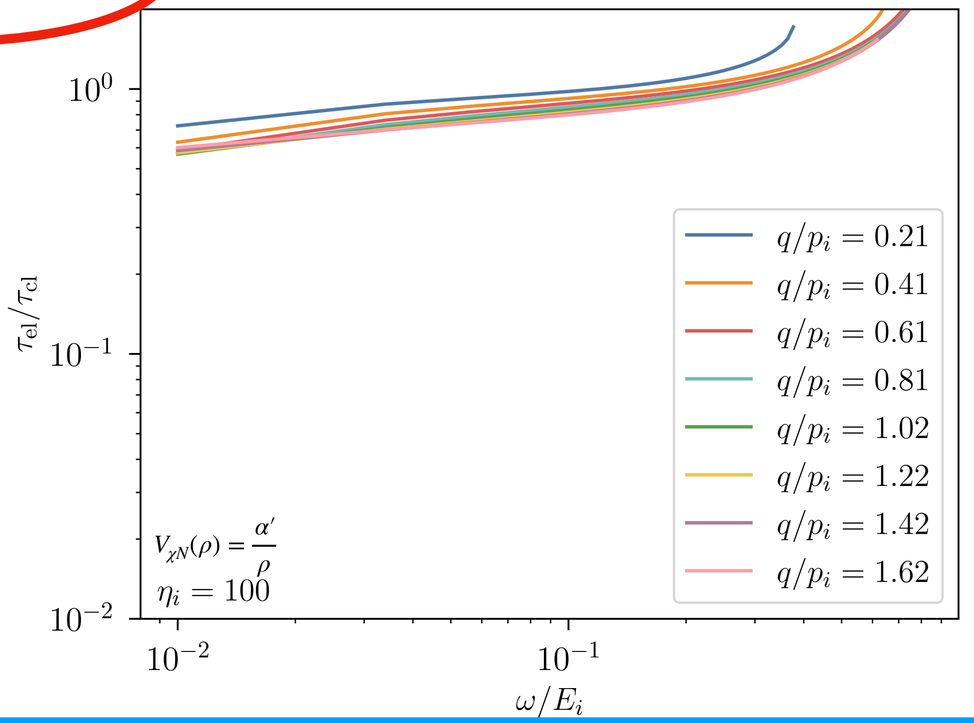
$$|\,\widetilde{\mathbf{F}}_{fi}(\omega, q, p_i))\,| = F_{\rm ad}(\omega/E_i, q/p_i)\, q\, |\,\mathscr{A}_{\rm el}(q, p_i)\,|$$

$|\,\widetilde{\mathbf{F}}_{fi}(\omega = 0, q, p_i)\,|/q$

Adiabatic form factor

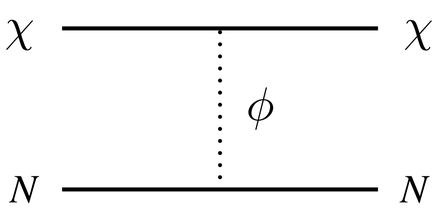
$$|\,F_{\rm ad}(\omega/E_i, q/p_i)\,|^2 = \exp\left[-\omega\,\tau_{\chi N}(\omega; q, p_i)\right]$$

Timescale



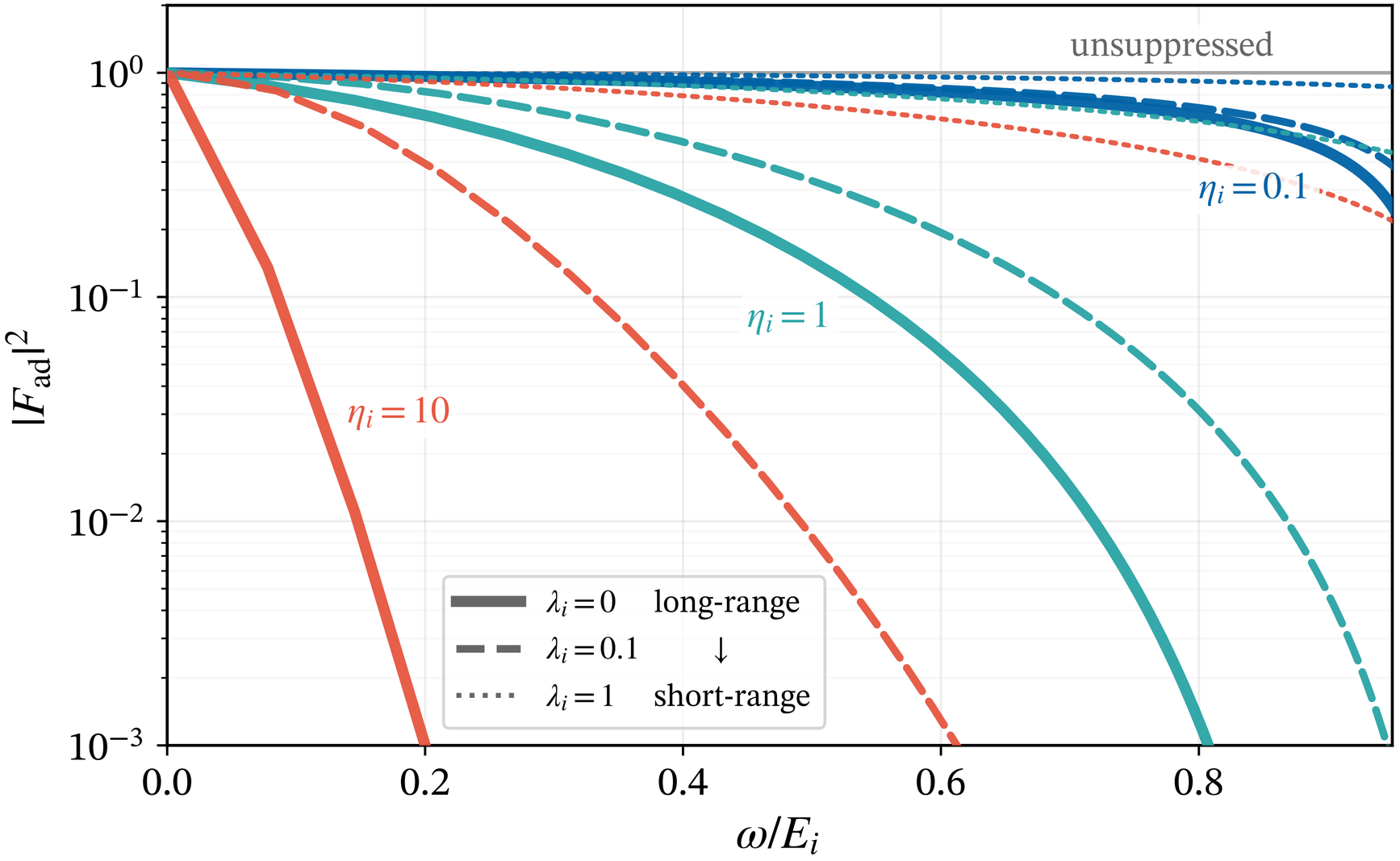
The adiabatic form factor - Yukawa scattering

$$V_{\chi N}(\rho) = \alpha' \frac{e^{-m_\phi \rho}}{\rho}$$



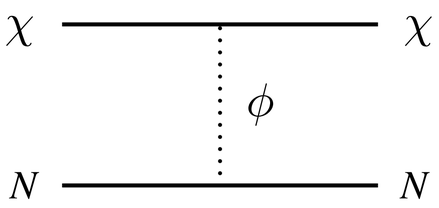
$$\eta = \alpha'/v_\rho$$

$$\lambda = m_\phi/p_\rho$$



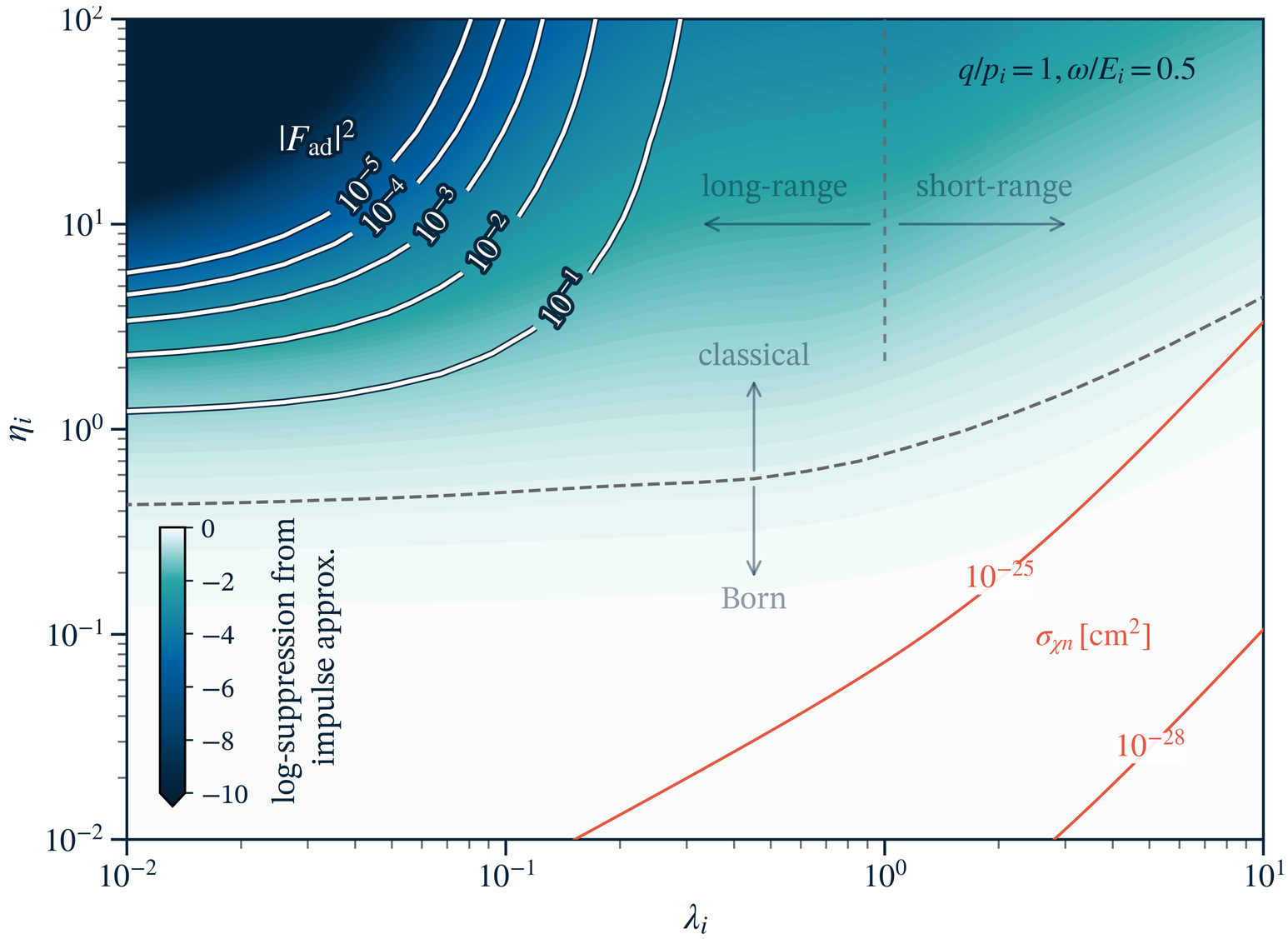
The adiabatic form factor - Yukawa scattering

$$V_{\chi N}(\rho) = \alpha' \frac{e^{-m_\phi \rho}}{\rho}$$



$$\eta = \alpha' / v_\rho$$

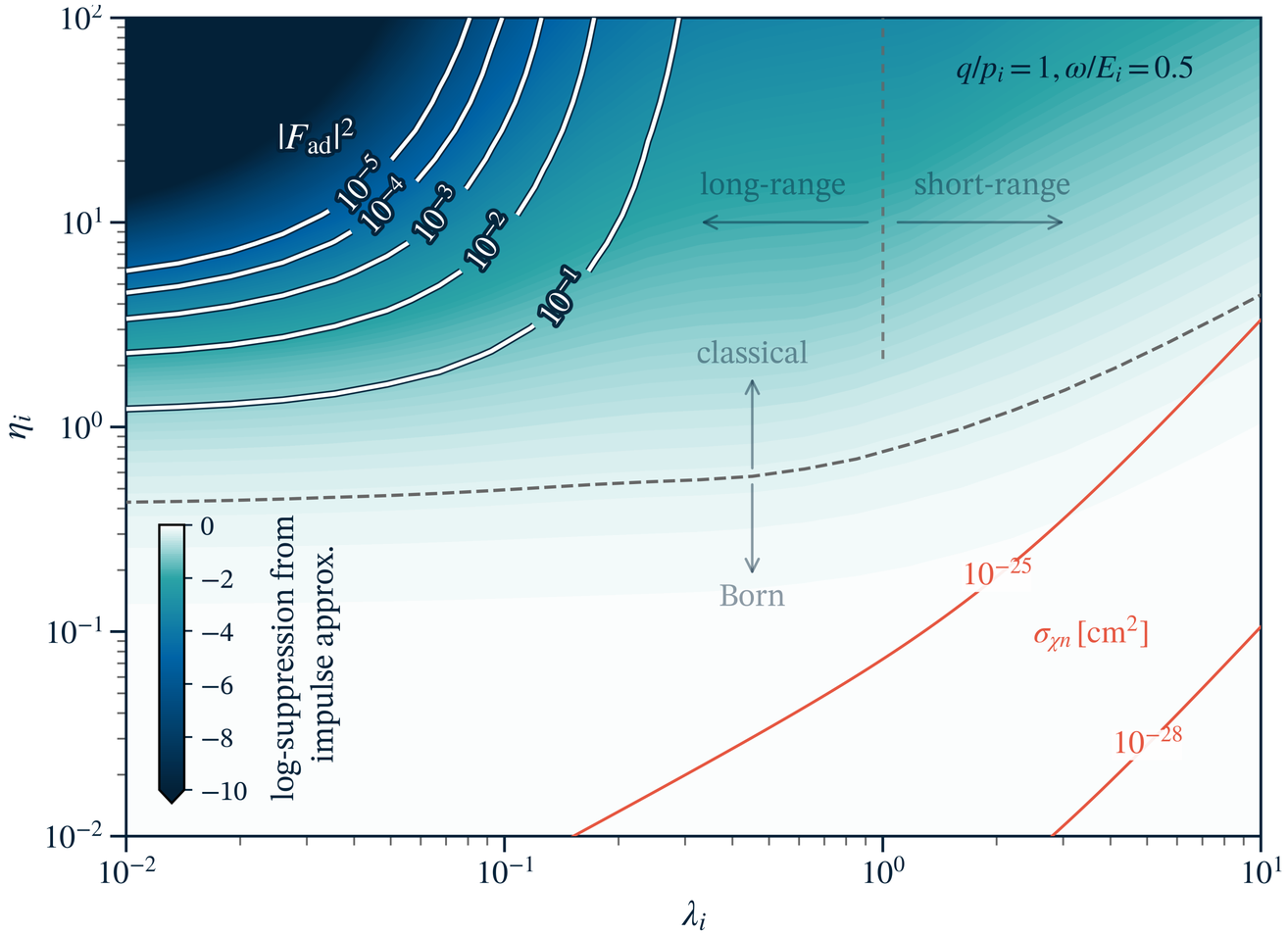
$$\lambda = m_\phi / p_\rho$$



The differential cross section

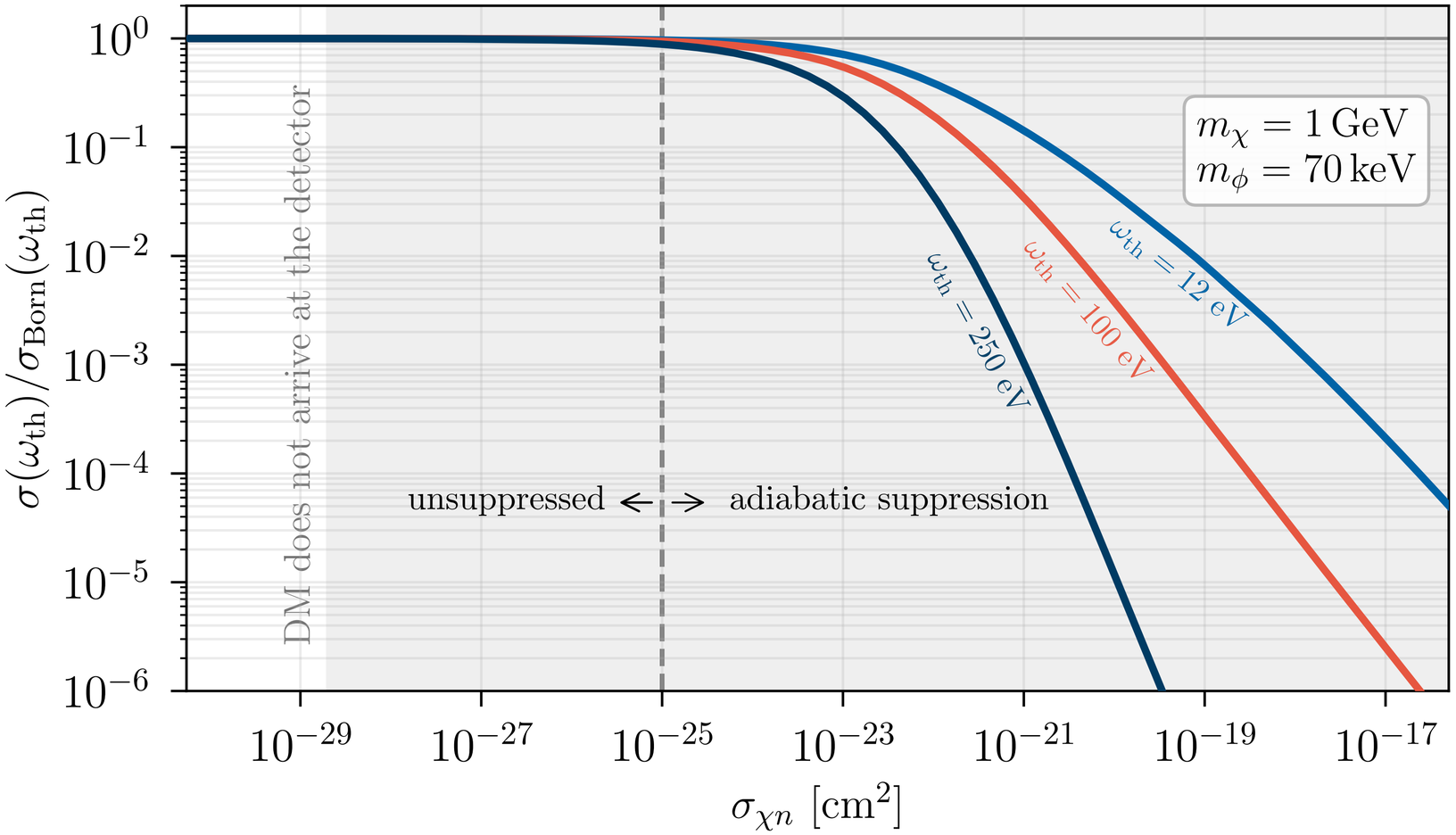
$$\frac{\mathrm{d}\sigma_{n\ell\to k}}{\mathrm{d}E_R\mathrm{d}\omega} = |F_{\mathrm{ad}}|^2 |F_{\mathrm{el}}|^2 \left.\frac{\mathrm{d}\sigma_{\chi N}}{\mathrm{d}E_R}\right|_{\mathrm{Born}} \frac{1}{2\pi}\frac{\mathrm{d}p_{n\ell\to k}}{\mathrm{d}\omega}$$

- We **always have elastic corrections** with respect to Born
- **Adiabatic corrections** present for **long-ranged** interactions
- Adiabatic suppression kills Migdal ionization



The total cross section

$$\sigma(\omega_{\rm th}) = \sum_{(n,\ell)} \int_{\max(\omega_{\rm th}, E_{n\ell})}^{p_i^2/2m_\chi} \int_{q_-^2/2m_N}^{q_+^2/2m_N} \mathrm{d}E_R \mathrm{d}\omega \, \frac{\mathrm{d}\sigma_{n\ell\to k}}{\mathrm{d}E_R \mathrm{d}\omega}$$



Concluding remarks and future prospects

- The impulse approximation is exact at Born level
- Temporal effects require **beyond Born** calculations
- **Adiabatic suppression emerges in long-range, semiclassical scattering**
- **DM searches lie in the unsuppressed regime**
- **Searches for Migdal with neutrons may get corrections**
- **Extend formalism to** other systems, such as **semiconductors**

Thank you for your time and attention!

Backup slides

Born = Impulse

$$\mathscr{M}_{fi} = (2\pi)^3 \ \widetilde{V}_{\chi N}(\mathbf{q}) \ \delta^3\Big(\mathbf{q} + \big(\mathbf{p}_{N,i} - \mathbf{p}_{N,f}\big)\Big) \langle \phi_{a_f}(\{\mathbf{x}'_{e,j}\}) \,|\, e^{i\frac{m_e}{m_N}(\mathbf{p}_{N,i} - \mathbf{p}_{N,f})\cdot \sum_j \hat{\mathbf{x}}'_{e,j}} \,|\, \phi_{a_i}(\{\mathbf{x}'_{e,j}\}) \rangle$$

The impulse approximation creeps up again

$$T(E) = V + VG_0^{(+)}(E)T(E) \qquad G_0^{+}(E) = (E - H_0 + i0^{+})^{-1}$$

$$T(E) = V + VG_0^{(+)}(E)V + VG_0^{(+)}(E)VG_0^{(+)}(E)V + VG_0^{(+)}(E)VG_0^{(+)}(E)VG_0^{(+)}(E)V + \ldots$$

Born approximation includes no propagators - ***instantaneous***

Calculating the Migdal rate - the impulse approximation

The electrons *can not* adjust to the recoiling nucleus during scattering

$$\langle \phi_f | \exp\left(i\beta \sum_{j=1}^{Z} \mathbf{q} \cdot \hat{\mathbf{x}}_j \right) | \phi_i \rangle$$

$$\frac{\mathrm{d}\sigma_{n\ell \to k}}{\mathrm{d}E_R \mathrm{d}\omega} = \frac{\mathrm{d}\sigma_{\chi N}}{\mathrm{d}E_R} \frac{1}{2\pi} \frac{\mathrm{d}p_{n\ell \to k}}{\mathrm{d}\omega}$$

In the impulse approximation, the **ionization probability multiplies the cross section**

Integrate cross section over DM distribution in lab frame

$$\frac{\mathrm{d}R}{\mathrm{d}E_R \mathrm{d}\omega} = \sum_{(n,\ell)} \frac{\rho_\chi}{m_\chi} \frac{m_N \sigma_{\chi n} A^2}{2\mu_{\chi n}^2} F_N^2(E_R) \frac{1}{2\pi} \frac{\mathrm{d}p_{n\ell \to k}}{\mathrm{d}\omega} \eta\left(v_{\min}(E_R, \omega) \right)$$

Energy lost to electrons

$$\frac{1}{\sqrt{2 m_N E_R}} \left(\frac{m_N E_R}{\mu_{\chi N}} + \omega \right)$$

M. Ibe, W. Nakano, Y. Shoji, and K. Suzuki, *Migdal Effect in Dark Matter Direct Detection Experiments*. JHEP 03, 194

DM direct detection - why do we care about Migdal?

Extreme loss of sensitivity below 1 GeV

$$\frac{\mathrm{d}R}{\mathrm{d}E_R} = \frac{\rho_\chi}{m_\chi}\frac{m_N \sigma_{\chi n} A^2}{2\mu_{\chi n}^2} F_N^2(E_R)\eta\left(v_{\min}(E_R)\right)$$

$$\sim \exp\left(-\frac{m_N E_R}{2\mu_{\chi N}^2 v_0^2}\right) \qquad \sqrt{\frac{m_N E_R}{2\mu_{\chi n}^2}}$$

Nobel gas detector have thresholds $\sim \mathcal{O}(\mathrm{keV})$

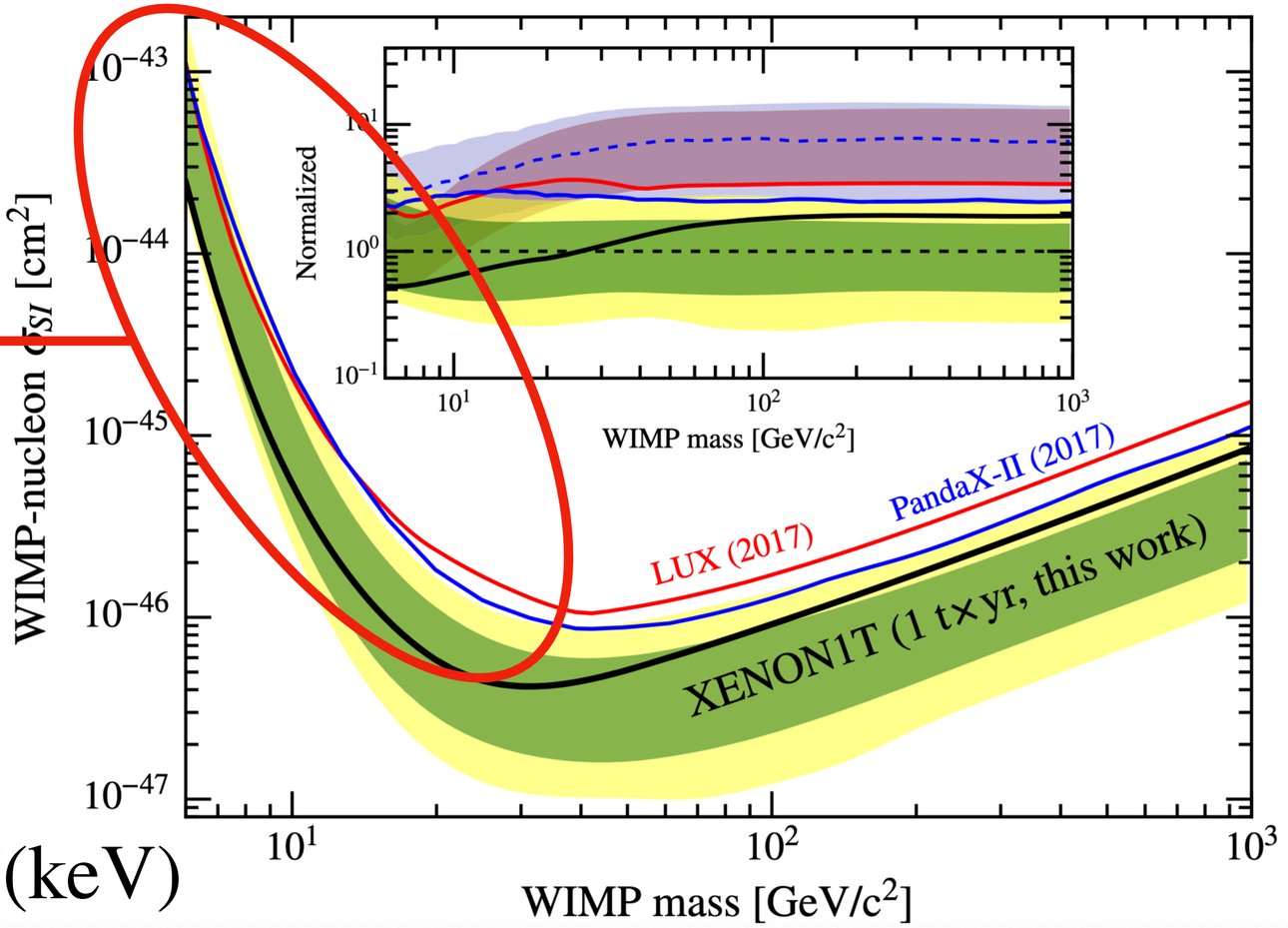


Figure from:
E. Aprile et al. (XENON), *Dark Matter Search Results from a One Ton-Year Exposure of XENON1T*, Phys. Rev. Lett. 121, 111302 (2018)

Computation of Yukawa force matrix elements

$$\psi^{\pm}_{\vec{p}_\rho}(\vec{\rho}) \simeq \sum_{\ell=0}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} C^{\pm}_{\ell m}(\vec{p}_\rho) R_{p_\rho \ell}(\rho) Y_{\ell m}(\hat{\vec{\rho}})$$

$$C^{\pm}_{\ell m}(\vec{p}_\rho) = \frac{4\pi i^{\ell} e^{\pm i\delta_\ell(p_\rho)}\, Y^*_{\ell m}(\hat{\vec{p}}_\rho)}{p_\rho}$$

$$\widetilde{\overrightarrow{F}}_{fi}(\omega, q, p_i) = -\sum_{\ell,\ell'} \sum_{m,m'} \frac{(2\ell'+1)(2\ell+1) i^{\ell-\ell'}}{p_i p_f} e^{i\left[\delta_\ell(p_i) + \delta_{\ell'}(p_f)\right]} \mathcal{J}^R_{\ell' p_f; \ell p_i} \langle \ell' m' | \hat{\hat{\rho}} | \ell m \rangle$$

$$\mathcal{J}^R_{\ell' p_f; \ell p_i} = \int_0^\infty \mathrm{d}\rho\, \rho^2 \frac{\mathrm{d}V_{\chi N}(\rho)}{\mathrm{d}\rho} R^*_{p_f \ell'}(\rho)\, R_{p_i \ell}(\rho)\,.$$

Using numerical radial continuum solutions of the Yukawa potential

Dipole selection rules and normalization

$$\ell' = \ell \pm 1, \quad m' = m, m \pm 1$$

The Coulomb force matrix element

$$\psi^{(\pm)}_{\vec{p}_\rho}(\vec{\rho}) = \Gamma(1 \pm i\eta) e^{-\frac{\pi}{2}\eta} e^{i\vec{p}_\rho \cdot \vec{\rho}} {}_1F_1(\mp i\eta, 1, \pm i p_\rho \rho - i\vec{p}_\rho \cdot \vec{\rho})$$

&

$$\widetilde{\vec{F}}_{fi}(\omega, q, p_i) = -\frac{(p_f^2 - p_i^2)^2}{4\mu_{\chi A}} \langle \psi^-_{\vec{p}_f} | \, \hat{\vec{\rho}} \, | \psi^+_{\vec{p}_i} \rangle$$

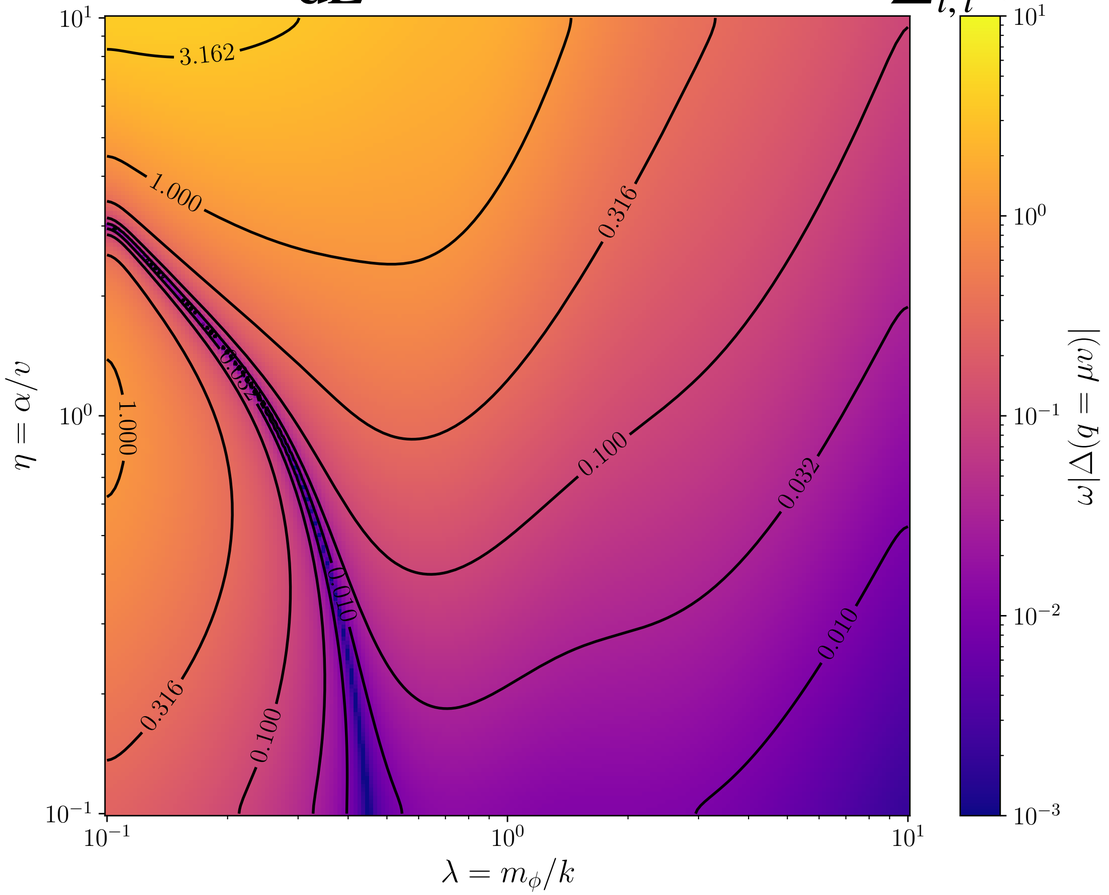
Transformed from gradient using dipole identities

$$\widetilde{\vec{F}}_{fi}(\omega, q, p_i) = \frac{4\pi e^{(\eta_i - \eta_f)\frac{\pi}{2}}}{\mu_{\chi A}(z-1)(p_i - p_f)^2} \Gamma(1 + i\eta_i)\Gamma(1 + i\eta_f) \left[(1-z)\frac{p_i - p_f}{p_i + p_f}\right]^{-i(\eta_i + \eta_f)} \left[\eta_i p_i \, {}_2F_1(z)(\vec{p}_f - \vec{p}_i) + i(1-z) {}_2F_1{}'(z)(p_i \vec{p}_f - p_f \vec{p}_i)\right]$$

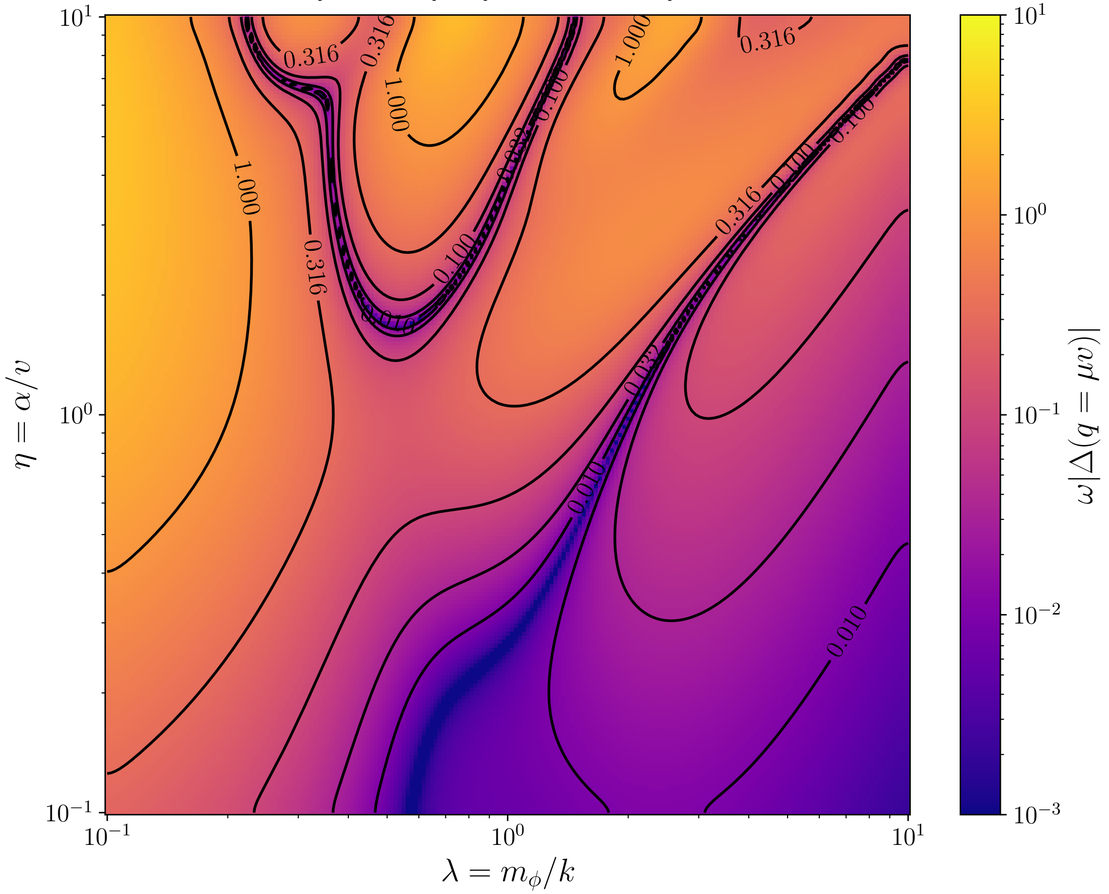
$${}_2F_1(z) = {}_2F_1(-i\eta_i, -i\eta_f, 1, z) \qquad {}_2F_1'(z) = \partial_z \, {}_2F_1(i\eta_i, i\eta_f, 1, z) = -\eta_i \eta_f \, {}_2F_1(i\eta_i + 1, i\eta_f + 1, 2, z) \qquad z = \frac{-2(p_i p_f - \vec{p}_i \cdot \vec{p}_f)}{(p_i - p_f)^2}$$

Elastic scattering timescales

$$\Delta = \frac{\mathrm{d}}{\mathrm{d}E}\log S \qquad \Delta(E,\theta) = \frac{\sum_{l,l'}(2l+1)(2l'+1)\sin(2\delta_l - \delta_{l'})\sin\delta_{l'}P_l(\cos\theta)P_{l'}(\cos\theta)\partial_E\delta_l}{\sum_{l,l'}(2l+1)(2l'+1)e^{i(\delta_l-\delta_{l'})}\sin\delta_l\sin\delta_{l'}P_l(\cos\theta)P_{l'}(\cos\theta)}$$



Repulsive



Attractive

A toy model - an external classical force

$$\widehat{H}_0 = \frac{\hat{\mathbf{p}}_N^2}{2m_N} + \sum_{j=1}^{Z} \frac{\hat{\mathbf{p}}_{e,j}^2}{2m_e} + \sum_{j=1}^{Z} V_{eN}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_N) + \frac{1}{2}\sum_{j\neq k} V_{ee}(\hat{\mathbf{x}}_{e,j} - \hat{\mathbf{x}}_{e,k})$$

$$\widehat{H}_e\,|\phi_a\rangle = E_a\,|\phi_a\rangle$$

Couple nucleus to external force

$$\widehat{V}(t) = -\,\mathbf{F}_N(t)\cdot\hat{\mathbf{x}}_N$$

$$\widehat{H}_{\text{eff}}(t) = \widehat{H}_0 - m_e\mathbf{a}(t)\cdot\sum_j \hat{\mathbf{r}}_j - m_A\mathbf{a}(t)\cdot\widehat{\mathbf{X}}_{\text{CoM}}$$

$$w\left(\frac{t}{\Delta}\right)\frac{v_0}{\Delta}\hat{\mathbf{n}}$$

The timescale Δ is related to the width of the acceleration profile

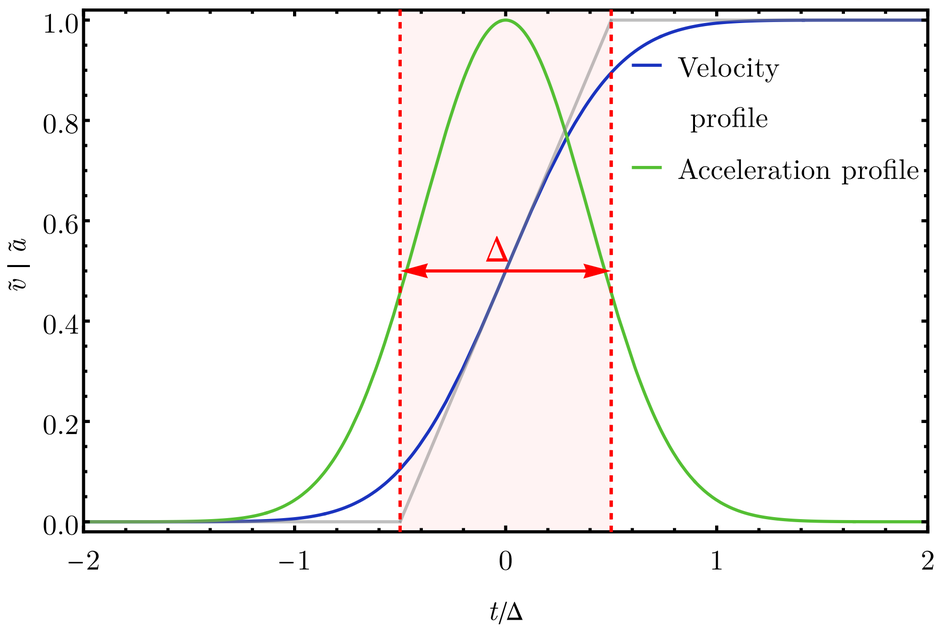
A toy model - an external classical force

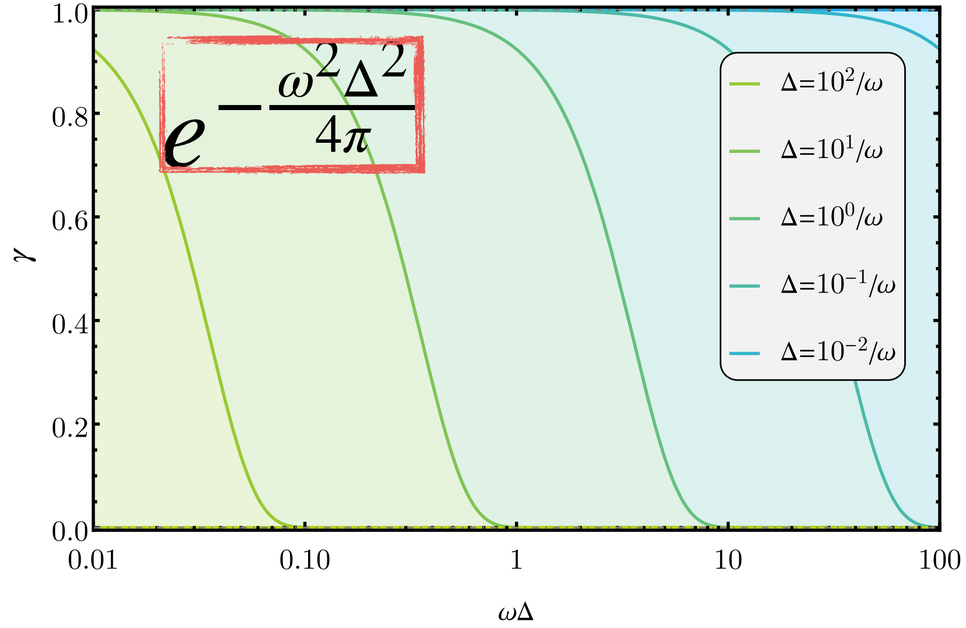
$$S_I = \mathcal{T} \exp\left(-i \int \mathrm{d}t\ \widehat{V}_I(t)\right)$$

We can directly compute in interaction picture

$$S_{fi}^{(1)} = -i\gamma(\omega\Delta) d_{ba} \longrightarrow d_{ba} = -\beta \langle \phi_a | \sum_j \mathbf{q} \cdot \hat{\mathbf{r}}_j | \phi_b \rangle$$

Adiabatic factor $\gamma(\omega\Delta) = \int \mathrm{d}\tau\, e^{-i\gamma\Delta\tau} w(\tau)$





A toy model - an external classical force

$$S_I = \mathscr{T} \exp\left(-i \int \mathrm{d}t \; \widehat{V}_I(t)\right)$$

We can directly compute in interaction picture

$$S_{fi}^{(n)} = \frac{(-i)^n}{n!} \prod_{k=1}^{n} \int \mathrm{d}o^{(k)} \, \gamma(\omega_{k-1,k}\Delta) d_{k-1,k} \delta(o^{(n)} - i)$$

Instantaneous $\Delta \to 0$

$$\gamma(\omega_{a,b}\Delta) \to 1$$

$$\lim_{\Delta \to 0} S = \exp\left(i\beta \sum_{j=1}^{Z} \mathbf{q} \cdot \hat{\mathbf{r}}_j\right)$$

Adiabatic $\Delta \to \infty$

$$\gamma(\omega_{a,b}\Delta) \to \delta(a - b)$$

Sum of terms becomes exponential

$$\lim_{\Delta \to \infty} S = \mathrm{diag} \exp\left(i\beta \sum_{j=1}^{Z} \mathbf{q} \cdot \hat{\mathbf{r}}_j\right)$$

Low-energy EFT from a QFT

- Local DM density $\rho \sim 0.3\,\mathrm{GeV/cm^3}$ and typical velocity is $v \sim 10^{-3}\,c$ → **non-relativistic**
- Typical momentum transfer $q \sim 10 - 100\,\mathrm{MeV}$ can **interact coherently with nucleus** when $1/q \gg 10\,\mathrm{fm}$

$$\mathscr{L}_{EFT} \supset \frac{C_V}{m_Z}\bar{\chi}\gamma^\mu\chi\bar{q}\gamma_\mu q + \ldots \rightarrow \langle n(k')\,|\,\bar{q}\gamma^\mu q\,|\,n(k)\rangle = \bar{Q}_n(k')\left[F_1^{Q,n}(q^2)\gamma^\mu + \frac{i}{2m_N}F_2^{Q,n}(q^2)\sigma^{\mu\nu}q_\nu\right]Q_n(k)$$

Z-scale DM-quark EFT

EFT with partons and form factors

$$\mathscr{M} = \frac{C_V}{m_Z^2}m_\chi \sum_{Q=u,d}\left(N_p m_p F_1^{Q,p}(\vec{q}^2) + N_n m_n F_1^{Q,n}(\vec{q}^2)\right)\bar{\zeta}_\alpha\zeta\bar{\xi}_\beta\xi_\beta \rightarrow \frac{\mathrm{d}R}{\mathrm{d}E_R\mathrm{d}v_\chi} = \frac{1}{m_N}\frac{\rho_\chi}{m_\chi}\frac{\mathrm{d}\sigma}{\mathrm{d}E_R}v_\chi\tilde{f}(v_\chi)$$

Matrix element can be used to match to NR Hamiltonian;
Fourier transform is effective NR potential

Differential event rate with astrophysics