Fermion mass relations from one-parameter models

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1. Introduction: the flavour puzzle
2. One-parameter models and mass relations
3. Running and threshold corrections
4. Outlook: higher-order groups and H6
5. Conclusions

The flavour puzzle

The Yukawa sector contains most of the SM parameters, with many redundancies. Gauge principles don't explain the rich structure observed in fermion masses and mixing.

PDG 2024

$$m_u : m_c : m_t \sim 10^{-6} : 10^{-3} : 1$$

$$m_d : m_s : m_b \sim 10^{-3} : 10^{-2} : 1$$

$$m_e : m_\mu : m_\tau \sim 10^{-4} : 10^{-2} : 1$$

Can we explain these hierarchies with minimal input parameters?

Crash course on Modular flavour

String-inspired: τ can describe the complex structure of a compact torus T^2.

Modular symmetry as a flavour symmetry (Feruglio, 2017)

$$G_{\text{mod}} \simeq SL(2,\mathbb{Z})/\ker\rho \text{ (Liu and Ding, 2022)}$$

Yukawa couplings become functions of the modulus

$$W \supset Y_{ij}^{(k)}(\tau)\psi_i\psi_j H$$

Proximity-induced hierarchies

$$\tau \simeq i\infty, \qquad q = e^{2\pi i\tau}, \qquad \epsilon \sim |q|^{1/N} \ll 1$$

$$m_1 : m_2 : m_3 \sim c_1\epsilon^p : c_2\epsilon^q : c_3$$

For more on modular flavour, see the talks by **Ferruccio Feruglio**, **Mu-Chun Chen** and **Arsenii Titov**.

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What is a One-Parameter Model (OPM)?

- In a generic modular flavour model, several invariant contractions can contribute to the same sector.

$$W_f \supset \sum_i \alpha_i \left(Y_{r_i}^{(k_i)}(\tau)\, \psi_f^c \psi_f\right)_1 H_f$$

- Each contraction comes with an independent coefficient α_i.
- We restrict ourselves to the most predictive possibility: one contraction per sector.

One-parameter limit

$$W_f = \alpha_f \left(Y_r^{(k)}(\tau)\, \psi_f^c \psi_f\right)_1 H_f$$

$$M_f(\tau) = \alpha_f v_f Y_r^{(k)}(\tau)$$

Key idea

The overall factor fixes m_3, the modulus fixes the ratios.

$$\frac{m_1}{m_3} = F_1(\tau), \qquad \frac{m_2}{m_3} = F_2(\tau).$$

Why this is already surprising

Single miracle

- A phenomenologically correct one-parameter mass matrix is already a small miracle.
- α_1 fixes the overall mass scale.
- ϵ fixes one mass ratio.
- The remaining ratio is predicted by group theory.

Double miracle

- Consider two OPMs sharing the same ϵ.
- The first sector fixes ϵ.
- In the second sector, α_2 only fixes the overall scale.
- Combining both sectors predicts three non-trivial mass relations.

One common modulus correlates independent flavour sectors.

One input gives four mass ratios!

Two promising hierarchies

- In our scan of finite modular groups, two hierarchies are especially suggestive.
- H_2 is close to the charged-lepton hierarchy.
- H_5 is close to the down-quark hierarchy.
- Both arise in $\Delta(384)$ and can share the same modulus.

$$\epsilon = |q|^{1/16}, \qquad q = e^{2\pi i \tau}.$$

Charged leptons

$$H_2: \quad m_e : m_\mu : m_\tau = 4\sqrt{2}\,\epsilon^6 : 2\epsilon^2 : 1$$

Down quarks

$$H_5: \quad m_d : m_s : m_b = 4\epsilon^5 : 2\sqrt{2}\,\epsilon^3 : 1$$

Double miracle realized

Same ϵ for two independent sectors.

High-scale mass relations

- The common ϵ can be eliminated.
- The output is a set of exact mass relations at $M_F \simeq 2 \times 10^{16}$ GeV.

Down-only

$$R_q: \quad 1 = \frac{1}{2\sqrt{2}} \frac{m_s^5}{m_b^2 m_d^3}$$

Lepton-only

$$R_\ell: \quad 1 = \frac{1}{\sqrt{2}} \frac{m_\mu^3}{m_\tau^2 m_e}$$

Inter-sector

$$R_i: \quad 1 = \frac{1}{\sqrt{2}} \frac{m_s^2}{m_b^2} \frac{m_\tau}{m_e}$$

Do these high-scale relations survive running and matching?

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A general consideration

- **Any** flavour model makes predictions at a flavour scale M_F, while the masses are measured at low energies.
- The RG evolution from M_F to M_Z depends on the UV completion.
- This matters especially in an extremely predictive model.

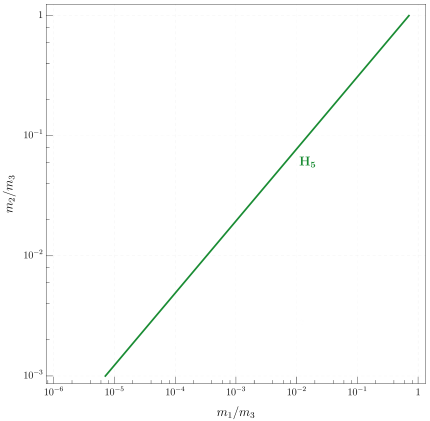
Minimal supersymmetric completion

Predictive modular flavour models rely on the holomorphicity of the superpotential. We therefore consider the MSSM.

$$M_F \xrightarrow{\text{MSSM running}} M_{\text{SUSY}} \xrightarrow[\text{matching at } M_{\text{SUSY}}]{\text{SM running}} M_Z$$

$$M_F \simeq 2 \times 10^{16}\,\text{GeV}, \qquad \tan\beta,\ M_{\text{SUSY}},\ \Delta_i$$

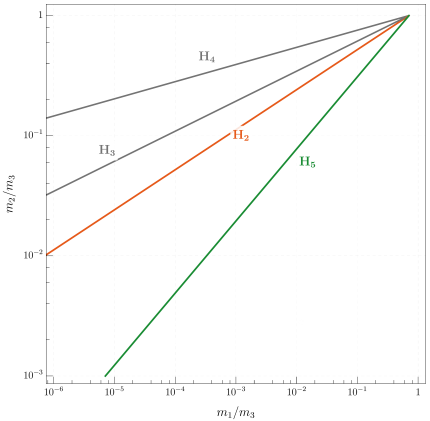
Hierarchy patterns



$$m_1 : m_2 : m_3$$

$$H_5 : \quad 4\epsilon^5 : 2\sqrt{2}\epsilon^3 : 1$$

Hierarchy patterns



$$m_1 : m_2 : m_3$$

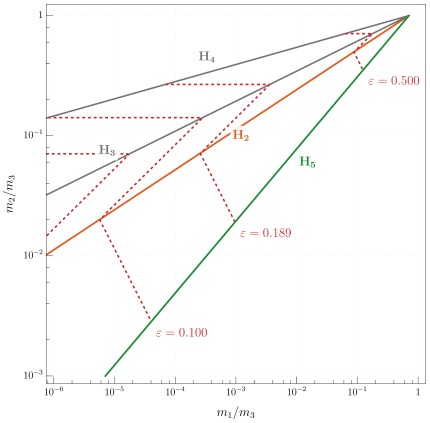
$$H_5: \quad 4\epsilon^5 : 2\sqrt{2}\epsilon^3 : 1$$

$$H_2: \quad 4\sqrt{2}\epsilon^6 : 2\epsilon^2 : 1$$

$$H_3: \quad 2\sqrt{2}\epsilon^4 : \sqrt{2}\epsilon : 1$$

$$H_4: \quad 8\epsilon^7 : \sqrt{2}\epsilon : 1$$

Hierarchy patterns



$$m_1 : m_2 : m_3$$

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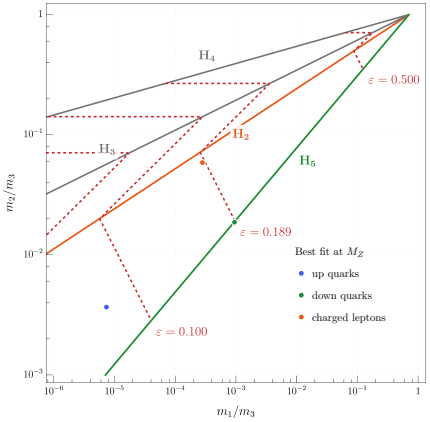
$$H_2: \quad 4\sqrt{2}\epsilon^6 : 2\epsilon^2 : 1$$

$$H_3: \quad 2\sqrt{2}\epsilon^4 : \sqrt{2}\epsilon : 1$$

$$H_4: \quad 8\epsilon^7 : \sqrt{2}\epsilon : 1$$

Same ϵ for all hierarchies.

Hierarchy patterns



$$m_1 : m_2 : m_3$$

H_5 : $4\epsilon^5 : 2\sqrt{2}\epsilon^3 : 1$

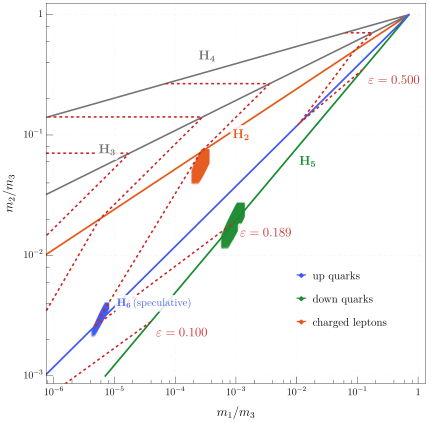
H_2 : $4\sqrt{2}\epsilon^6 : 2\epsilon^2 : 1$

H_3 : $2\sqrt{2}\epsilon^4 : \sqrt{2}\epsilon : 1$

H_4 : $8\epsilon^7 : \sqrt{2}\epsilon : 1$

Experimental data at M_Z.

Hierarchy patterns



$$m_1 : m_2 : m_3$$

H_5 : $4\epsilon^5 : 2\sqrt{2}\epsilon^3 : 1$

H_2 : $4\sqrt{2}\epsilon^6 : 2\epsilon^2 : 1$

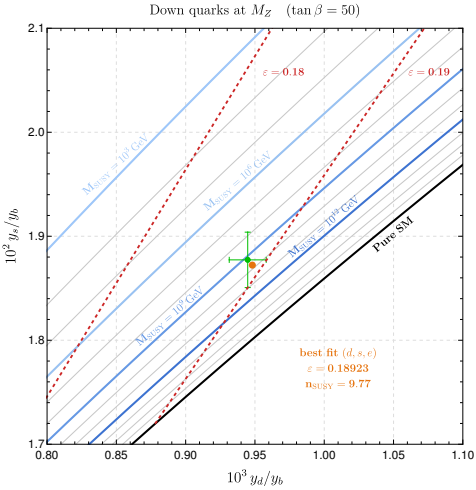
H_3 : $2\sqrt{2}\epsilon^4 : \sqrt{2}\epsilon : 1$

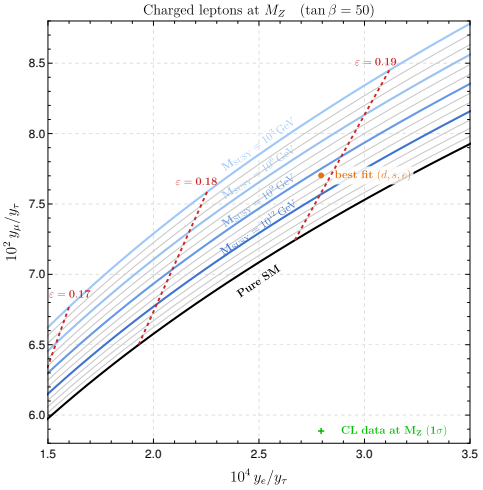
H_4 : $8\epsilon^7 : \sqrt{2}\epsilon : 1$

Running and threshold corrections

- Run $M_Z \to M_F$ with **REAP (Antusch et al.)**.
- Scan M_F, $M_{\rm SUSY}$ and $\tan\beta$.
- Threshold-like corrections: up to 15% for down quarks and charged leptons, 5% for up quarks.

Fit before thresholds





Required threshold corrections

- We do not want arbitrary thresholds.
- The successful down ratios should remain unchanged.
- The successful e/τ ratio should remain unchanged.
- Only μ/τ should be shifted.

$$\left.\frac{y_{\ell_i}}{y_\tau}\right|_{\rm SM} = \kappa_i \left.\frac{y_{\ell_i}}{y_\tau}\right|_{\rm MSSM}$$

Threshold ansatz

$$\Delta_d = \Delta_s = \Delta_b$$

$$\Delta_e = \Delta_\tau, \qquad \Delta_\mu \neq \Delta_\tau$$

Required matching

$$\kappa_i = \frac{1+\Delta_i}{1+\Delta_\tau}$$

$$\kappa_e = 1, \qquad \kappa_\mu \simeq 0.764$$

A soft SUSY ansatz for selective thresholds

Standard one-loop electroweak SUSY threshold corrections, in the large-$\tan\beta$ regime, neglecting trilinear terms.

Soft SUSY spectrum

$$M_1 = M_2 = \mu_H \equiv M$$

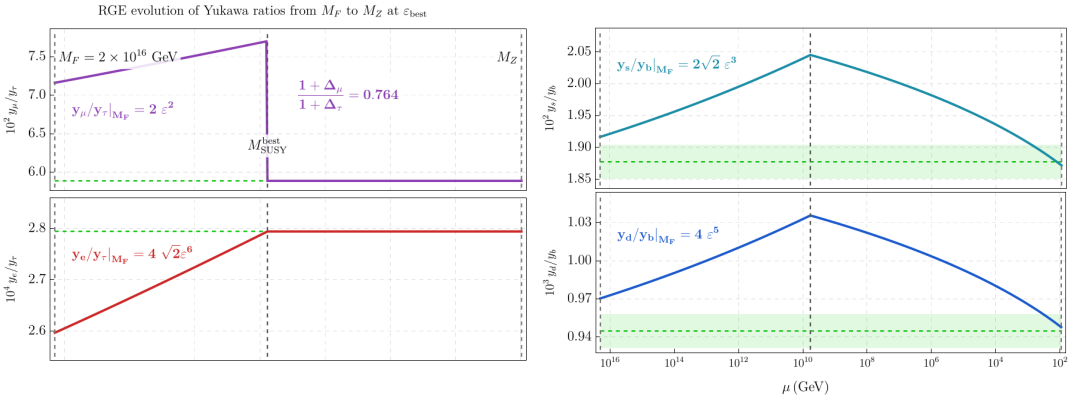
$$m_{e_1} = m_{L_1} = m_{e_3} = m_{L_3} = x_{e\tau} M$$

$$m_{e_2} = m_{L_2} = x_\mu M$$

Resulting slepton mass hierarchy

$$\frac{m_{\tilde{\mu}_{L,R}}}{m_{\tilde{e}_{L,R}}} = \frac{m_{\tilde{\mu}_{L,R}}}{m_{\tilde{\tau}_{L,R}}} \simeq 41$$

Running with μ thresholds



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A hidden relation in the OPM hierarchies

With the notation

$$m_1 : m_2 : m_3 = A\epsilon^p : B\epsilon^q : 1,$$

the hierarchies found so far satisfy

$$\frac{(m_2/m_3)^{p/q}}{m_1/m_3} = \sqrt{2}.$$

- This holds for $H_1, \ldots, H_5$ and for all the additional (unpublished) OPMs found so far in higher-order groups.
- The new cases include groups unrelated to $\Delta(6N^2)$.
- Open question: is this relation generic to OPMs?

A hint from the up sector: an H6 candidate

- If we assume the hidden relation and apply it to the up-quark sector, the data is very close to $p/q = 2$.
- The corresponding hierarchy is

$$H_6: \qquad m_u : m_c : m_t = 4\sqrt{2}\,\epsilon^8 : 2\sqrt{2}\,\epsilon^4 : 1.$$

- We have now found an explicit candidate with H_6 in the up sector and H_5 in the down sector, but can't fit CKM.

Work in progress.

A speculative nine-mass relation

If the charged leptons, down quarks and up quarks are assigned to H_2, H_5 and H_6, respectively, and share the same ϵ, one obtains (amongst many others) a relation involving all charged fermions:

$$\boxed{\frac{1}{2\sqrt{2}}\left(\frac{m_e m_\mu}{m_\tau^2}\right)\left(\frac{m_d m_s}{m_b^2}\right)\left(\frac{m_t^3}{m_u m_c^2}\right)=1}$$

At the electroweak scale, using the current central values and uncertainties,

$$\frac{1}{2\sqrt{2}}\left(\frac{m_e m_\mu}{m_\tau^2}\right)\left(\frac{m_d m_s}{m_b^2}\right)\left(\frac{m_t^3}{m_u m_c^2}\right)\Bigg|_{M_Z}=1.05\pm0.05.$$

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Conclusions

- OPMs are highly constrained. Our published scan found five hierarchy patterns, $H_1, \ldots, H_5$.
- $H_2 \oplus H_5$ gives a viable toy model for charged leptons and down quarks. High weight up quark assignment fits CKM (no prediction).
- Further OPMs in higher-order groups, including groups outside the $\Delta(6n^2)$ family, satisfy the same *magic relation*.
- A new $H_6 \oplus H_5$ candidate reproduces the quark mass hierarchies, but predicts a (very) wrong CKM.

Thank you!

Based on

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Questions welcome!

Published search: five patterns (four inequivalent)

- Published result: scan of all groups of order < 100.
- The first non-trivial hierarchy appears in $\Delta(96)$.
- Enlarging to $\Delta(384) \supset \Delta(96)$ gives the patterns shown here.
- Recent (unpublished) work: More OPMs in higher-order groups, including groups unrelated to $\Delta(6N^2)$.

Hierarchy	Spectrum	Group
H_1	$4\sqrt{2}\,\epsilon^3 : 2\epsilon : 1$	$\Delta(96)$
H_2	$4\sqrt{2}\,\epsilon^6 : 2\epsilon^2 : 1$	$\Delta(384)$
H_3	$2\sqrt{2}\,\epsilon^4 : \sqrt{2}\,\epsilon : 1$	$\Delta(384)$
H_4	$8\,\epsilon^7 : \sqrt{2}\,\epsilon : 1$	$\Delta(384)$
H_5	$4\,\epsilon^5 : 2\sqrt{2}\,\epsilon^3 : 1$	$\Delta(384)$

The coefficients are exactly fixed by group theory.

Modular group and finite quotients

The modular group is generated by

$$S : \tau \mapsto -\frac{1}{\tau}, \qquad T : \tau \mapsto \tau + 1.$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \qquad \gamma\tau = \frac{a\tau + b}{c\tau + d}.$$

$$\Gamma(N) = \{\gamma : \gamma \equiv \mathbb{I} \pmod{N}\}.$$

Ordinary finite modular groups

$$\Gamma'_N = SL(2,\mathbb{Z})/\Gamma(N).$$

$$\Gamma_N \simeq \Gamma'_N/\{\pm\mathbb{I}\}, \qquad N > 2.$$

N	Γ_N	Γ'_N
2	S_3	S_3
3	A_4	T'
4	S_4	S'_4
5	A_5	A'_5

More general finite modular quotients

Finite-image representations

$$\rho : SL(2,\mathbb{Z}) \longrightarrow GL(V), \qquad G_{\rm mod} = SL(2,\mathbb{Z})/\ker\rho \simeq {\rm Im}\,\rho.$$

The kernel need not be a principal congruence subgroup $\Gamma(N)$.

Liu and Ding (2022)

A different generalized construction

$$G_{\rm mod}(N', N'') = \Gamma(N')/\Gamma(N''), \qquad N' \mid N''.$$

$$G_{\rm mod}(1, N'') = \Gamma'_{N''}.$$

Li, Liu and Ding (2021)

Matter fields and modular forms

Matter multiplets

$$\psi \longrightarrow (c\tau + d)^{-k_\psi} \rho_\psi(\gamma)\psi.$$

- k_ψ is the modular weight.
- ρ_ψ describes the flavour multiplet.

Vector-valued modular forms

$$Y^{(k)}(\gamma\tau) = (c\tau + d)^k \rho_Y(\gamma) Y^{(k)}(\tau).$$

- For fixed weight and representation, the space of modular forms is finite-dimensional.

Superpotential

$$W \supset \left(Y_r^{(k)}(\tau)\, \psi_i \psi_j H\right)_{\mathbf{1}}, \qquad k = k_{\psi_i} + k_{\psi_j} + k_H.$$

Proximity-induced hierarchies

- At special values of τ, residual symmetries remain.
- Near those points, residual symmetries are slightly broken.
- Couplings vanishing at the symmetric point can reappear as powers of a small displacement.

$$\tau \simeq i\infty, \qquad q = e^{2\pi i \tau}.$$

$$\epsilon \sim |q|^{1/N} \ll 1.$$

Mass-matrix expansion

$$M_f(\tau) = M_f^{(0)} + \epsilon M_f^{(1)} + \epsilon^2 M_f^{(2)} + \cdots .$$

Generic spectrum

$$m_1 : m_2 : m_3 \sim c_1\epsilon^p : c_2\epsilon^q : c_3.$$

OPM search strategy

- Scan finite modular groups with triplet irreps.
- Fix candidate triplets for ψ_f^c and ψ_f.
- Check the rank at the symmetric point.
- Require exactly one modular form and one invariant contraction.
- Compute the resulting mass spectrum.

Typical failure modes

- No suitable triplet pair.
- Several invariant contractions or modular forms.
- Rank greater than one at the symmetric point.
- One or more generations remain massless.

$$G_{\text{mod}} \Rightarrow (\rho_{\psi^c}, \rho_{\psi}) \Rightarrow Y_r^{(k)} \Rightarrow M_f(\tau) \Rightarrow m_1 : m_2 : m_3$$

$\Delta(384)$ realization

- Charged leptons and down quarks use different sextet modular forms.
- Both sectors are controlled by the same modulus.
- The Higgs field is a trivial singlet with $k_H = 0$.

$$k_{d^c} + k_Q = 2, \qquad (\rho_{d^c}, \rho_Q) \sim (3_7, 3_8).$$

$$k_{e^c} + k_L = 2, \qquad (\rho_{e^c}, \rho_L) \sim (3_1, 3_2).$$

$$W_d = \alpha_d (Y^{(2)}_{6_6} d^c Q)_{\mathbf{1}} H_d, \qquad W_e = \alpha_e (Y^{(2)}_{6_0} e^c L)_{\mathbf{1}} H_d.$$

Vector-valued modular forms in $\Delta(384)$

$$q = e^{2\pi i \tau}, \qquad \epsilon = |q|^{1/16}.$$

Charged leptons: $Y^{(2)}_{6_0}$

$$Y^{(2)}_{6_0}(\tau) = \begin{pmatrix} -8q^{3/8}(1+\cdots) \\ 2q^{1/8}(1+\cdots) \\ 4q^{1/2}(-1+\cdots) \\ 1+\cdots \\ -4q^{5/8}(3+\cdots) \\ -16q^{7/8}(1+\cdots) \end{pmatrix}$$

Down quarks: $Y^{(2)}_{6_6}$

$$Y^{(2)}_{6_6}(\tau) = \begin{pmatrix} 2\sqrt{2}\,q^{11/16}(5+\cdots) \\ 2\sqrt{2}\,q^{3/16}(1+\cdots) \\ -2q^{1/2}(1+\cdots) \\ -1+\cdots \\ 4\sqrt{2}\,q^{5/16}(1+\cdots) \\ -4\sqrt{2}\,q^{13/16}(3+\cdots) \end{pmatrix}$$

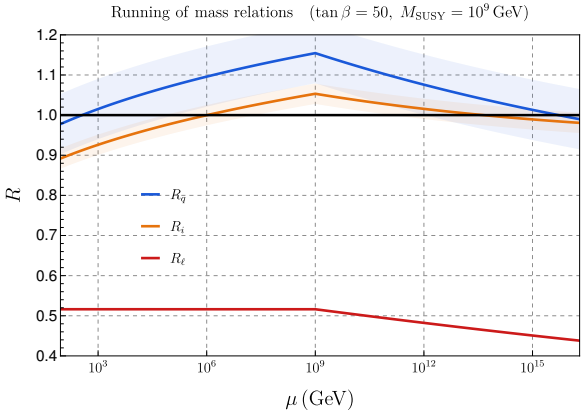
Mass matrices

$$M_e(\tau) = \alpha_e v_d \begin{pmatrix} 0 & \sqrt{2}\, Y^{(2)}_{6_0,5} & -\sqrt{2}\, Y^{(2)}_{6_0,2} \\ \sqrt{2}\, Y^{(2)}_{6_0,4} & Y^{(2)}_{6_0,6} & Y^{(2)}_{6_0,1} \\ -\sqrt{2}\, Y^{(2)}_{6_0,3} & -Y^{(2)}_{6_0,1} & -Y^{(2)}_{6_0,6} \end{pmatrix}$$

$$M_d(\tau) = \alpha_d v_d \begin{pmatrix} -\sqrt{2}\, Y^{(2)}_{6_6,3} & 0 & \sqrt{2}\, Y^{(2)}_{6_6,4} \\ Y^{(2)}_{6_6,5} & -\sqrt{2}\, Y^{(2)}_{6_6,1} & -Y^{(2)}_{6_6,6} \\ -Y^{(2)}_{6_6,6} & -\sqrt{2}\, Y^{(2)}_{6_6,2} & Y^{(2)}_{6_6,5} \end{pmatrix}$$

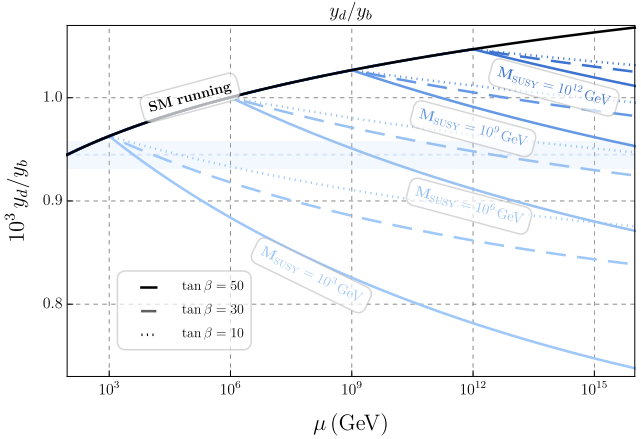
Near $i\infty$, these matrices give H_2 and H_5.

First diagnostic: run the relations

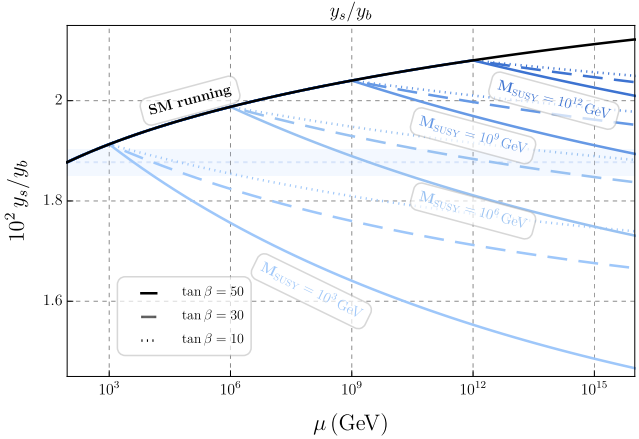


- Start from the experimental Yukawas at M_Z and run to M_F.
- Compare with the high-scale predictions $R_q = R_i = R_\ell = 1$.
- R_q and R_i work extremely well. R_ℓ fails.
- The mismatch is isolated in y_μ/y_τ.

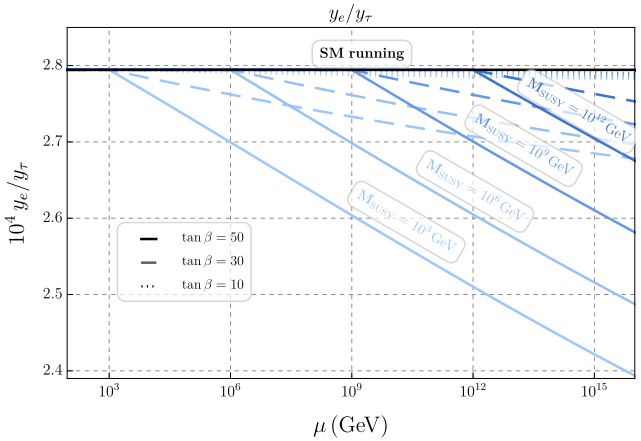
MSSM running: y_d/y_b



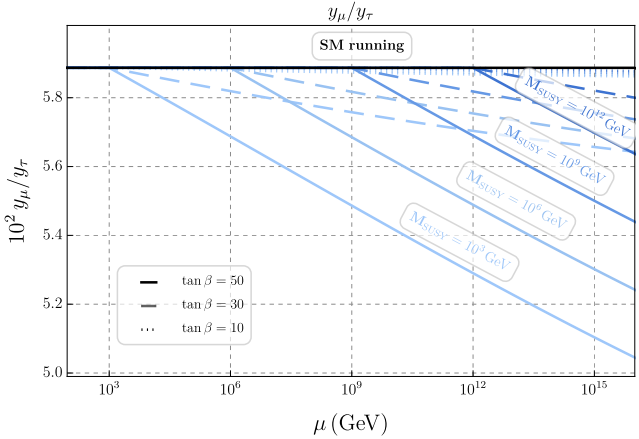
MSSM running: y_s/y_b



MSSM running: y_e/y_τ



MSSM running: y_μ/y_τ



Charged-lepton threshold formulae

$$\Delta_i = \frac{\tan\beta}{16\pi^2}\left(\tau_i^W + \tau_i^B\right).$$

$$\tau_i^W = \frac{3}{2} g_2^2 \frac{M_2}{\mu_H} h_2\left(\frac{M_2^2}{\mu_H^2}, \frac{m_{L_i}^2}{\mu_H^2}\right).$$

$$\tau_i^B = \frac{3}{5} g_1^2 \left[-\frac{\mu_H}{M_1} h_2\left(\frac{m_{e_i}^2}{M_1^2}, \frac{m_{L_i}^2}{M_1^2}\right) + \frac{M_1}{\mu_H} h_2\left(\frac{m_{e_i}^2}{\mu_H^2}, \frac{M_1^2}{\mu_H^2}\right) - \frac{1}{2}\frac{M_1}{\mu_H} h_2\left(\frac{M_1^2}{\mu_H^2}, \frac{m_{L_i}^2}{\mu_H^2}\right)\right].$$

$$h_2(x,y) = \frac{x\log x}{(1-x)(x-y)} + \frac{y\log y}{(1-y)(y-x)}.$$

Selective charged-lepton thresholds: benchmark

- Large $\tan\beta$ and negligible trilinear terms.
- First and third charged-lepton generations have identical corrections.

$$M_1 = M_2 = \mu_H \equiv M.$$

$$m_{e_1} = m_{L_1} = m_{e_3} = m_{L_3} \equiv x_{e\tau} M, \qquad m_{e_2} = m_{L_2} \equiv x_\mu M.$$

$$g_1(M_{\rm SUSY}^{\rm best}) = 0.515, \qquad g_2(M_{\rm SUSY}^{\rm best}) = 0.5703, \qquad \tan\beta = 50.$$

$$x_\mu = 0.5, \qquad x_{e\tau} = 0.012137.$$

$$\frac{1+\Delta_\mu}{1+\Delta_\tau} \simeq 0.764, \qquad \frac{1+\Delta_e}{1+\Delta_\tau} = 1.$$