Eclectic Flavour Symmetries without Flavons

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in collaboration with:

Antonio Marrone, 2609.16118 – to appear in JHEP



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The flavour puzzle



flavour symmetry ?

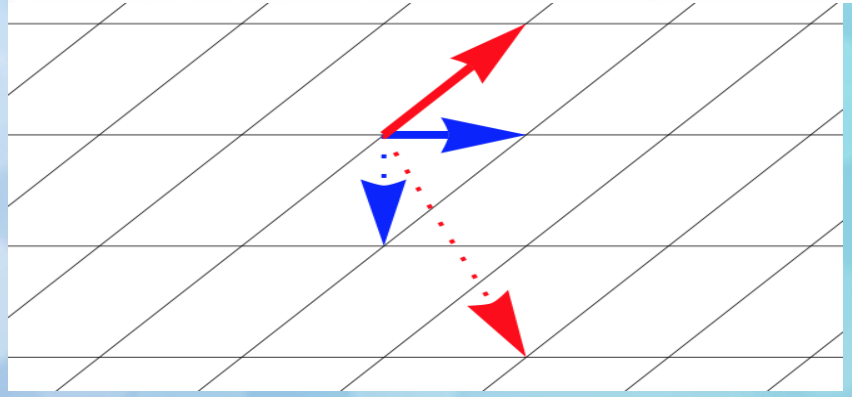
The flavour symmetry approach

modular flavour symmetries $G_{mod} = SL(2,Z)$

$$\tau \xrightarrow{\gamma} \frac{a\tau + b}{c\tau + d}\,, \qquad \Phi_I \longmapsto (c\tau + d)^{-k_I} \rho_I(\gamma) \Phi_I\,,$$

finite image of $SL(2,Z)$: $S_3, A_4, T', S_4, \ldots$

A torus is $\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$: a lattice in $\mathbb{C}$ with different bases (same lattice).



lattice left invariant by modular transformations:

The flavour symmetry approach

modular flavour symmetries $G_{mod} = SL(2,Z)$

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finite image of $SL(2,Z)$: $S_3, A_4, T', S_4, \ldots$

traditional flavour symmetries G_{trad}

$$\tau \xrightarrow{g} \tau\,, \qquad \Phi_I \longmapsto U_I(g) \Phi_I\,,$$

traditional flavour symmetry ➡ flavons ➡ alignment problem

Flavons = new scalar fields whose vacuum values break the flavour symmetry.

A real plague for model building:

- ✗ extra fields and parameters
- ✗ complicated (and *ad hoc*) potentials
- ✗ vacuum alignment problem
- ✗ loss of predictivity



Eclectic flavour groups

in TD approaches, e.g. string theories, they both appear

heterotic string compactification containing a T^2/Z_3 sector

[Niles, Ramos-Sánchez, Vaudrevange, arXiv:2001.01736
Nilles, Ramos-Sánchez, Vaudrevange, arXiv:2004.05200
Baur, Nilles, Ramos-Sánchez, Trautner, Vaudrevange, arXiv:2112.06940
Trautner, arXiv:2204.06581]

$U_I(g)$ $\rho_I(\gamma)$

$$\Omega(1) \cong \Delta(27) \rtimes T'$$

eclectic flavour group of order 648

[can be extended to include also CP and R-symmetries]

modular + traditional flavour symmetry are not independent

eclectic flavour symmetry → flavons → alignment problem

related approach: quasi-eclectic flavour symmetries
[Chen, Knapp-Pérez, Ramos-Hamud, Ramos-Sánchez, Ratz, Shukla, arXiv:2108.02240]

mass matrix of matter fields ϕ and ϕ^c transforming as $3^{(0)}$ of $\Omega(1)$

$$M_{\rm ecl}(\tau,\varphi) = c \begin{pmatrix} \hat{Y}_2\,\varphi_1 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_3 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_2 \\ -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_3 & \hat{Y}_2\,\varphi_2 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_1 \\ -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_2 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_1 & \hat{Y}_2\,\varphi_3 \end{pmatrix},$$

[Baur, Nilles, Ramos-Sánchez, Trautner, Vaudrevange, arXiv:2207.10677]

$\widehat{\mathbf{Y_2}}(\tau) = (\hat{Y}_1, \hat{Y}_2)^T$ doublet of modular forms of T'

$\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ flavon triplet of $\Delta(27)$

Here is the problem. The modular symmetry removed flavons and eclectic symmetry has brought them back.

can we eliminate flavons?

Can "traditional" flavour symmetry also have a geometric origin?

representation		finite image		infinite modular group
$\rho_I(\gamma)$	$\rightarrow$	T'	$\rightarrow$	$SL(2,\mathbb{Z})$
$U_I(g)$	$\rightarrow$	$\Delta(27)$	$\rightarrow$	?

can we eliminate flavons?

Can "traditional" flavour symmetry also have a geometric origin?

representation		finite image		infinite modular group
$\rho_I(\gamma)$	$\rightarrow$	T'	$\rightarrow$	$SL(2,\mathbb{Z})$
$U_I(g)$	$\rightarrow$	$\Delta(27)$	$\rightarrow$	$\mathrm{H}(\mathbb{Z})$

$SL(2,Z)$ and H act on two moduli (τ, z) such that

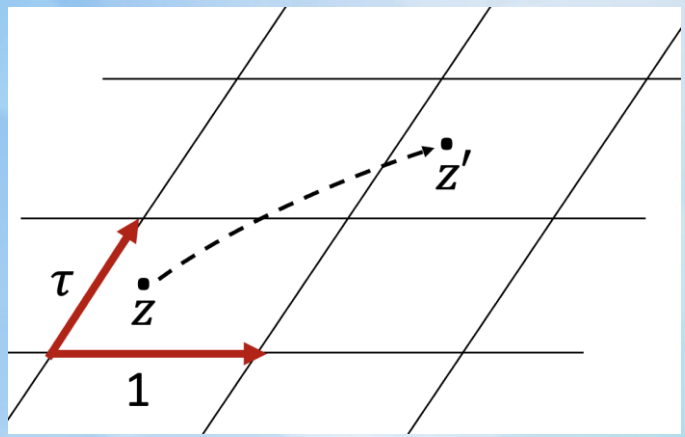
	τ	z
$SL(2,\mathbb{Z})$	$\gamma\,\tau$	$\gamma\,z$
$\mathrm{H}(\mathbb{Z})$	τ	$h\,z$

modular and "traditional" transformations are now parts of one infinite symmetry

H broken by the new modulus z: no flavons needed

candidate for $\mathrm{H}(\mathbb{Z})$

action on moduli (τ, z) - simplest choice: a gauge transformation



$$z \xrightarrow{h} z + \lambda\,\tau + \mu \qquad (\lambda, \mu) \in \mathbb{Z}$$

X Y

$X\,Y = Y\,X$ on moduli

X and Y remind translations for a charged particle in a magnetized torus

$X\,Y = Y\,X$

on coordinates

$X\,Y = e^{i\,2\pi\,\phi} Y\,X$ (Z^2)

on the particle wave function

Heisenberg integral group

$$\mathrm{H}(\mathbb{Z}) = \langle X, Y, Z \,|\, Z\ central, X\,Y = Z^2\,Y\,X \rangle$$

	τ	z
$SL(2,\mathbb{Z})$	$\dfrac{a\,\tau + b}{c\,\tau + d}$	$\dfrac{z}{c\,\tau + d}$
$\mathrm{H}(\mathbb{Z})$	τ	$z + \lambda\,\tau + \mu$

determined by

holomorphicity

$\gamma\, H\, \gamma^{-1} = H$

$\gamma\, \emptyset = \emptyset$

integral Jacobi group $\quad \Gamma^J \cong \mathrm{H}(\mathbb{Z}) \rtimes SL(2,\mathbb{Z})$

image mod 3 $\quad \Omega(1) \cong \Delta(27) \rtimes T'$

level-3 SUSY Jacobi-invariant theory

define an automorphy factor,
generalizing $J_k(\gamma,\tau) = (c\tau + d)^k$ for $SL(2,\mathbb{Z})$

$$w \equiv \frac{z}{3}$$

$$J_{k,m}(g;\tau,w) = (c\tau + d)^k e^{i\,2\pi\, m\, \theta(g;\tau,w)}$$

$$m \in 3\mathbb{Z}\,.$$

Jacobi index

- Jacobi modular forms

$$Y_{k,m}(g\cdot(\tau,w)) = J_{k,m}(g;\tau,w)\,\rho(\widetilde{g})\,Y_{k,m}(\tau,w)\,,$$

- matter multiplets under Jacobi group

$$\Phi_I \longmapsto J_{k_I,m_I}(g;\tau,w)^{-1}\rho_I(\widetilde{g})\,\Phi_I\,.$$

a minimal charged-fermion assignment

(k, m, ρ)

$q, l, u^c, d^c, e^c \sim (1{,}6, 3^{(0)})$

$h_{u,d} \sim (0,0,1)$

$$Y_f = y_f \left(\mathcal{M}_{DG} + r_f \mathcal{M}_{EF} \right), \qquad f = u, d, e\,.$$

$$\mathcal{M}_{YX} = \begin{pmatrix} Y_2 X_1 & -\frac{Y_1}{\sqrt{2}} X_3 & -\frac{Y_1}{\sqrt{2}} X_2 \\ -\frac{Y_1}{\sqrt{2}} X_3 & Y_2 X_2 & -\frac{Y_1}{\sqrt{2}} X_1 \\ -\frac{Y_1}{\sqrt{2}} X_2 & -\frac{Y_1}{\sqrt{2}} X_1 & Y_2 X_3 \end{pmatrix}$$

no flavons

(X_1, X_2, X_3) (Y_1, Y_2) Jacobi modular forms

same (τ, w) for all charged fermions

compare with conventional eclectic

$$M_{\rm ecl}(\tau,\varphi) = c \begin{pmatrix} \hat{Y}_2\,\varphi_1 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_3 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_2 \\ -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_3 & \hat{Y}_2\,\varphi_2 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_1 \\ -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_2 & -\frac{\hat{Y}_1}{\sqrt{2}}\,\varphi_1 & \hat{Y}_2\,\varphi_3 \end{pmatrix},$$



$$\begin{pmatrix} Y_2\,X_1 & -\frac{Y_1}{\sqrt{2}}\,X_3 & -\frac{Y_1}{\sqrt{2}}\,X_2 \\ -\frac{Y_1}{\sqrt{2}}\,X_3 & Y_2\,X_2 & -\frac{Y_1}{\sqrt{2}}\,X_1 \\ -\frac{Y_1}{\sqrt{2}}\,X_2 & -\frac{Y_1}{\sqrt{2}}\,X_1 & Y_2\,X_3 \end{pmatrix}$$

a geometrical origin of the hierarchies

- at the two-torsion locus $\eta = w - \frac{a\,\tau + b}{6} = 0$ $\quad a, b \in \mathbb{Z}$

- a basis exists where

$$Y_f(\tau, w) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$$

$$m_u = m_d = m_e = 0$$

$$V_{us} = V_{ub} = J_{CKM} = 0$$

"modular protection"

- a small displacements $\eta \neq 0$ from this exact locus generates first-generation masses, remaining mixing and CP violation

CP at the Lagrangian level

r_u, r_d, r_e real

$\langle \tau \rangle \neq -\langle \tau \rangle^*$

$\langle w \rangle \neq -\langle w \rangle^*$

generic Jacobi vacuum

$J_{CKM} \neq 0$

spontaneous CP violation

holomorphy is not enough

universal matter metric

$K \propto 1$

does not reproduce the quark data

symmetry-selected kinetic tensors

$$K = 1 + \sum_Y c_Y \, Y \, Y^+ + \cdots$$

no new flavour spurions

quarks + leptons: proof of principle

$$\Phi_I \to \varepsilon(g)^{p_I} J_{k_I,m_I}(g;\tau,w)^{-1}\,\rho_I(g)\,\Phi_I$$

$q, l, u^c, d^c, e^c, \nu^c \sim$ (1,6, triplets) × multiplier

$$\chi^2_{q+\ell} = 2.0520, \qquad N_{\rm par} = 13, \qquad {\rm dof} = +2.$$

NO

$$\sum_i m_i = 0.202\ eV$$

$m_\beta = 0.06\ eV$

$m_{\beta\beta} = 0.06\ eV$

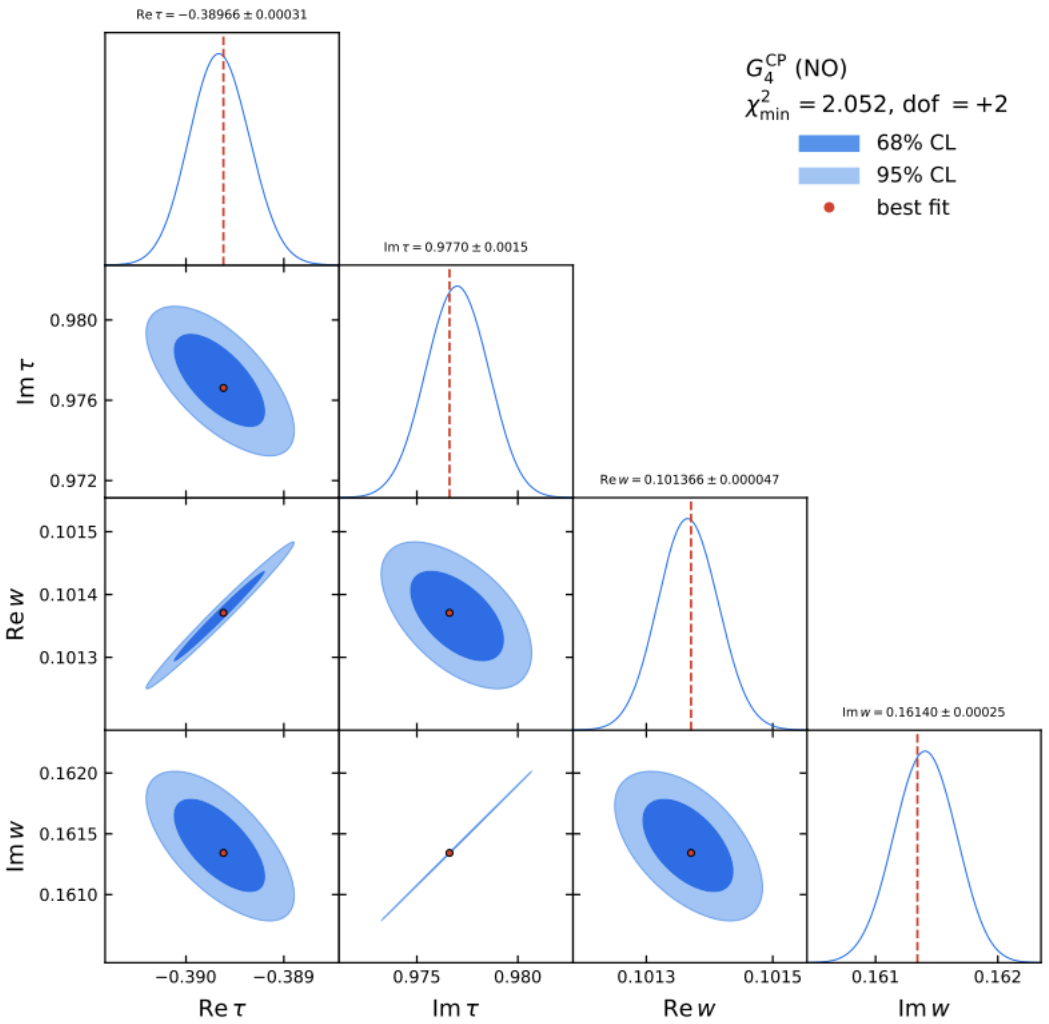
Experiment	Isotope	$m_{\beta\beta}$ (90% C.L.)
KamLAND-Zen 800 (current)	^{136}Xe	**< 28–122 meV**
LEGEND-200 (current)	^{76}Ge	**< 75–200 meV**
CUPID (future)	^{100}Mo	**9.6–16.3 meV**
LEGEND-1000 (future)	^{76}Ge	**9–19 meV**
nEXO (future)	^{136}Xe	**4.7–20.3 meV**

$sin^2\theta_{23} = 0.460$

$\delta^l_{CP}/\pi = 1.46$ not used in the fit

important caveat: some preferred Kähler metrics are strongly hierarchical

The fit rediscovers the geometry



$$\eta = w - \frac{\tau + 1}{6} = 0$$

best fit: $|\eta| \cong 1.5 \times 10^{-3}$

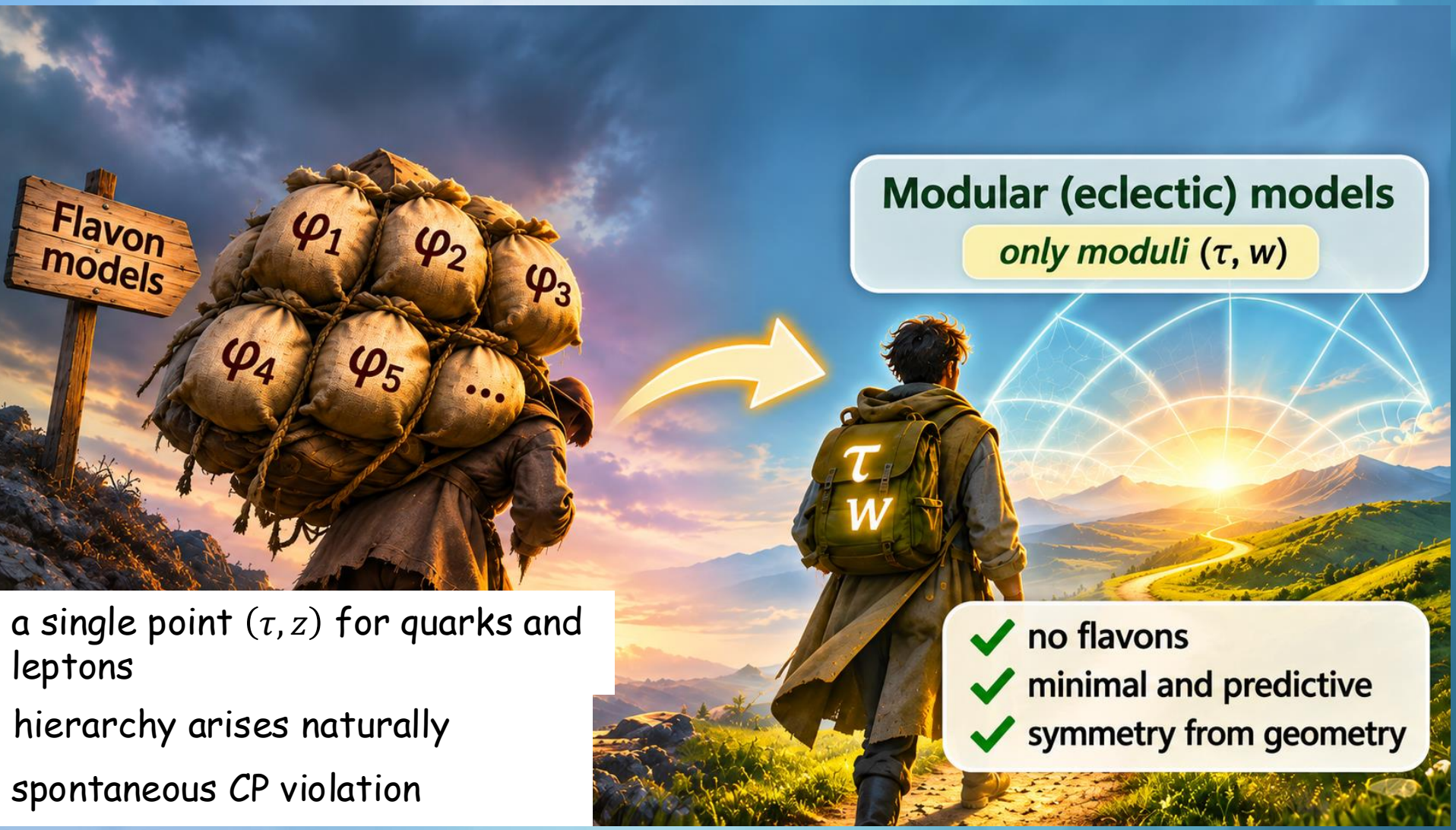
the best fit lies very close to the two-torsion locus $\eta = 0$ (geometry not imposed in the fit)

The fit rediscovers the geometrical relation $w = \frac{\tau+1}{6}$

Conclusion

modular and "traditional" transformations can be incorporated into one infinite symmetry

$$\Gamma^J \cong \mathrm{H}(\mathbb{Z}) \rtimes SL(2,\mathbb{Z})$$



a single point (τ, z) for quarks and leptons

hierarchy arises naturally

spontaneous CP violation

proof of principle for a realistic model of quark and leptons


THANK
YOU!

back-up slides

$T^2/\mathbb{Z}_K$

K	Integral parent and coordinates	Finite remnants
3	$\mathrm{H}(\mathbb{Z}) \rtimes \mathrm{SL}(2,\mathbb{Z})$ acting on (τ, z)	Traditional: $H_3(\mathbb{F}_3) \simeq \Delta(27)$ Modular: $\mathrm{SL}(2,\mathbb{F}_3) \simeq T'$
4	$\mathrm{H}(\mathbb{Z}) \rtimes \mathrm{SL}(2,\mathbb{Z})$ acting on (τ, z)	Traditional: $\mathcal{P}_1 \simeq [16,13]$ Modular: $\mathrm{SL}(2,\mathbb{F}_2) \simeq S_3$
2	$H_5(\mathbb{Z}) \rtimes \left[(\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U) \rtimes \mathbb{Z}_2^{\mathrm{mir}}\right]$ acting on $(T, U; z_T, z_U)$	Traditional: $H_5(\mathbb{F}_2) \simeq [32,49]$ Modular: $(S_3^T \times S_3^U) \rtimes \mathbb{Z}_2^{\mathrm{mir}}$

the construction is not a trick tailored only to $\Omega(1)$

$T^2/\mathbb{Z}_6$

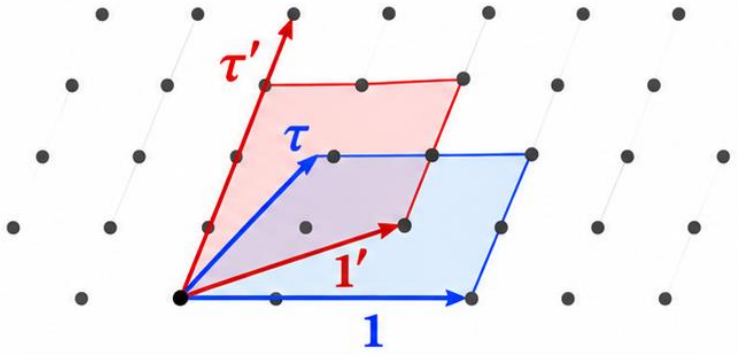
structurally
different
no analogous
semidirect product

Jacobi as a gauge group

Geometric view of the *Jacobi group*

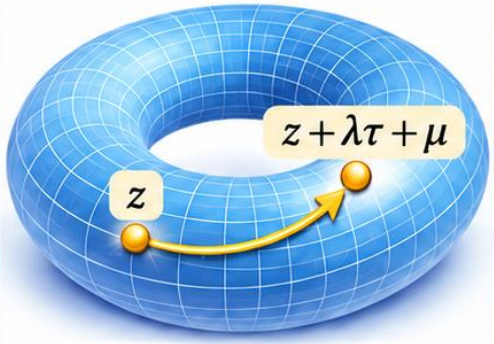
Torus and modular symmetry

A torus is $\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$: a lattice in $\mathbb{C}$ with different bases (same lattice).



$$\tau \longrightarrow \tau' = \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

Lattice translations (on the torus)



same point on the torus

Jacobi action on (τ, z)

$$(\tau, z) \to \left(\frac{a\tau+b}{c\tau+d}, \frac{z+\lambda\tau+\mu}{c\tau+d} \right)$$

Jacobi group = redundancy in describing the torus and a marked point (τ, z).

Jacobi automorphy factor

$$J_{k,m}(g;\tau,w) = (c\tau+d)^k \exp\left\{2\pi i m\left[\frac{c\,(w+(\lambda\tau+\mu)/3)^2}{c\tau+d} - \frac{\lambda^2\tau+6\lambda w+\kappa}{9}\right]\right\},$$

Kähler potential

$$K = K_{mod}(\tau, w) + K_{matter}$$

$$K_{mod}(\tau, w) = -h \log Im\,\tau + 4\pi\,\nu\,\frac{(Im\,w)^2}{Im\,\tau} \qquad h, \nu > 0$$

$$K_{matter} = \sum_{\phi} Im\,\tau^{-k_\phi}\, e^{4\pi\, m_\phi \frac{(Im\,w)^2}{Im\,\tau}}\, \phi^{+}\phi + K_{non-min}$$

$$K_{mod}(\tau, w) \xrightarrow{g} K_{mod}(\tau, w) + \log J_{h,\nu}(g; \tau, w) + \log \overline{J_{h,\nu}(g; \tau, w)}$$

superpotential

$$W = \sum Y_{I_1 \dots I_n}(\tau, w)\ \phi_{I_1} \dots \phi_{I_n}$$

assume W transform covariantly

$$W \xrightarrow{g} J_{k_W, m_W}\ \rho_W(g)\ W$$

invariance of the theory

$$k_W = m_W = 0$$
$$\rho_W = 1$$

rigid SUSY

$$k_W = -h \qquad m_W = -\nu$$
ρ_W unconstrained

local SUSY

$$Y_{I_1 \dots I_n}(\tau, w)\ \phi_{I_1} \dots \phi_{I_n}$$
not vanishing

$$k_W = k_Y - \sum_a k_{I_a} \qquad m_W = m_Y - \sum_a m_{I_a}$$

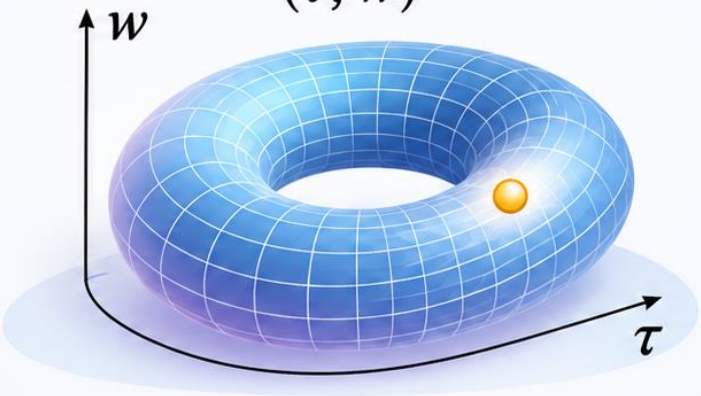
$$\rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \supset \rho_W$$

From Jacobi to *conventional eclectic symmetry*

Jacobi (symplectic) framework

Flavour symmetry acting on two moduli

(τ, w)

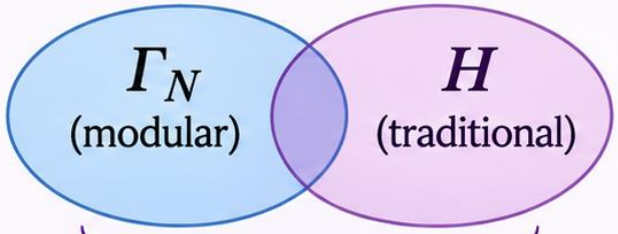


Unified flavour symmetry
from geometry

$w \to 0$

Conventional eclectic symmetry

Flavour symmetry from
modular + traditional groups



$\Gamma_N \rtimes H$

Standard eclectic symmetry
(as in previous models)

Takeaway: fixing $w = 0$ reduces the Jacobi framework to the conventional eclectic symmetry.

quarks + leptons: proof of principle

$$\Phi_I \to \varepsilon(g)^{p_I} J_{k_I,m_I}(g;\tau,w)^{-1}\, \rho_I(g)\, \Phi_I$$

$q, l, u^c, d^c, e^c, \nu^c \sim$ (1,6, triplets) × multiplier

Observable	$G_4^{\rm CP}$	Experimental input/reference
$\delta m^2\,[\mathrm{eV}^2]$	7.48×10^{-5}	$(7.48 \pm 0.20) \times 10^{-5}$
$\Delta m^2\,[\mathrm{eV}^2]$	2.495×10^{-3}	$(2.495 \pm 0.020) \times 10^{-3}$
$\sin^2\theta_{12}$	0.3086	0.3085 ± 0.0114
$\sin^2\theta_{13}$	0.02230	0.0223 ± 0.00062
$\sin^2\theta_{23}$	0.460	0.473 ± 0.024
δ^ℓ_{CP}/π	1.46	$[1.20 \pm 0.21]$ not used in the fit
$J_{\rm PMNS}$	-0.0333	–
$\sum_i m_i\,[\mathrm{eV}]$	0.202	–
$m_\beta\,[\mathrm{eV}]$	0.0616	–
$m_{\beta\beta}\,[\mathrm{eV}]$	0.0606	–

$$\chi^2_{q+\ell} = 2.0520, \qquad N_{\rm par} = 13, \qquad \mathrm{dof} = +2.$$

CP violation entirely from the moduli

important caveat: some preferred Kähler metrics are strongly hierarchical

good global agreement, but not yet a sharply predictive model

back-up: size of the Kähler deformations

$$\mathrm{cond}\, K_f = \frac{\lambda_{\max}(K_f)}{\lambda_{\min}(K_f)}.$$

charged-fermion reference fit

$$\mathrm{cond}\, K_{u^c} \simeq 7.1 \times 10^2,$$
$$\mathrm{cond}\, K_{d^c} \simeq 4.2.$$

global CP fit

$$\mathrm{cond}\, K_q \simeq 2.3 \times 10^5,$$
$$\mathrm{cond}\, K_{u^c} \simeq 3.6 \times 10^3.$$

main naturalness caveat: the successful fit is not a perturbatively small deformation of a universal metric:

Cfr. [Baur, Nilles, Ramos-Sánchez, Trautner, Vaudrevange, arXiv:2207.10677]

$cond\ K_u \approx 1.7 \times 10^4$ $cond\ K_d \approx 2.2 \times 10^2$ $cond\ K_q \approx 62$

best-fit parameters for G_4^{CP}

$$\tau = -0.389617 + 0.976616\,\mathrm{i}, \qquad w = 0.101371 + 0.161342\,\mathrm{i},$$
$$r_e = -2.52120, \qquad r_D = -2.99655,$$
$$r_u = 2.57839, \qquad r_d = 43.9479,$$
$$\kappa_G^q = 2.32347 \times 10^5, \qquad \kappa_G^{u^c} = -0.999722,$$
$$\kappa_F^{d^c} = 5.13287 \times 10^4, \qquad \kappa_G^{d^c} = 1.44257 \times 10^3,$$
$$k_\nu = 0.00439046\ \mathrm{eV}.$$

lepton observables at best-fit point

Observable	G_2	G_4	G_4^{CP}	Experimental input/reference
$\delta m^2\,[\mathrm{eV}^2]$	7.48×10^{-5}	7.48×10^{-5}	7.48×10^{-5}	$(7.48 \pm 0.20) \times 10^{-5}$
$\Delta m^2\,[\mathrm{eV}^2]$	2.495×10^{-3}	2.495×10^{-3}	2.495×10^{-3}	$(2.495 \pm 0.020) \times 10^{-3}$
$\sin^2\theta_{12}$	0.3085	0.3085	0.3086	0.3085 ± 0.0114
$\sin^2\theta_{13}$	0.02230	0.02230	0.02230	0.0223 ± 0.00062
$\sin^2\theta_{23}$	0.461	0.465	0.460	0.473 ± 0.024
δ^{ℓ}_{CP}/π	1.47	1.48	1.46	$[1.20 \pm 0.21]$ not used in the fit
J_{PMNS}	-0.0334	-0.0336	-0.0333	–
$\sum_i m_i\,[\mathrm{eV}]$	0.214	0.214	0.202	–
$m_\beta\,[\mathrm{eV}]$	0.0661	0.0660	0.0616	–
$m_{\beta\beta}\,[\mathrm{eV}]$	0.0654	0.0652	0.0606	–

Table 7: Neutrino observables at the representative normal-ordering best-fit points of G_2, G_4 and G_4^{CP}. The leptonic Dirac phase and absolute neutrino-mass observables are reported in the final rows as predictions evaluated at the same best-fit points. The experimental column shows the inputs used in the numerical fit from the global analyses in Refs. [71, 72]. The bracketed Dirac phase is shown as an experimental reference and is not used in the fit.

Low-weight Jacobi forms

weight 1 Jacobi doublets

$$D(\tau) \equiv \mathbf{Y}^{(1,0)}(\tau) = \begin{pmatrix} D_1(\tau) \\ D_2(\tau) \end{pmatrix} = \begin{pmatrix} -3\sqrt{2}\,\dfrac{\eta^3(3\tau)}{\eta(\tau)} \\ 3\,\dfrac{\eta^3(3\tau)}{\eta(\tau)} + \dfrac{\eta^3(\tau/3)}{\eta(\tau)} \end{pmatrix} .$$

$$\vartheta_{m,r}(\tau,w) = \sum_{n\in\mathbb{Z}} q^{(2mn+r)^2/(4m)} \xi^{2mn+r} , \qquad r \pmod{2m} .$$

For $m = 9$ introduce

$$A_9(\tau,w) = \vartheta_{9,0}(\tau,w) + \vartheta_{9,6}(\tau,w) + \vartheta_{9,12}(\tau,w) ,$$
$$B_9(\tau,w) = \vartheta_{9,3}(\tau,w) + \vartheta_{9,9}(\tau,w) + \vartheta_{9,15}(\tau,w) ,$$

and the index-three theta constants

$$C_r(\tau) = \vartheta_{3,r}(\tau,0) , \qquad r = 0,1,2,3 ,$$

where $\vartheta_{3,4}(\tau,0) = C_2(\tau)$ and $\vartheta_{3,5}(\tau,0) = C_1(\tau)$. An explicit basis of $\mathcal{J}^{(3)}_{1,9}$ is

$$E(\tau,w) \equiv \mathbf{Y}^{(1,9)}(\tau,w) = \begin{pmatrix} E_1(\tau,w) \\ E_2(\tau,w) \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\,[C_1(\tau)B_9(\tau,w) + C_2(\tau)A_9(\tau,w)] \\ C_0(\tau)A_9(\tau,w) + C_3(\tau)B_9(\tau,w) \end{pmatrix} .$$

weight 1 Jacobi triplets

$$\vartheta(\tau,w)=\sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2/2}\,\xi^{n+1/2}\,,\qquad q=e^{2\pi i\tau}\,,\qquad \xi=e^{2\pi i w}\,. \tag{193}$$

The theta blocks of Ref. [75] are

$$Q_{a,b}(\tau,w)=\frac{\vartheta(\tau,aw)\,\vartheta(\tau,bw)\,\vartheta(\tau,(a+b)w)}{\eta(\tau)}\,, \tag{194}$$

with weight one and classical index $m=a^2+ab+b^2$. The two seeds needed here are

$$q_3(\tau,w)\equiv Q_{1,1}(\tau,w)=\frac{\vartheta(\tau,w)^2\vartheta(\tau,2w)}{\eta(\tau)}\,, \tag{195}$$

$$q_{12}(\tau,w)\equiv Q_{2,2}(\tau,w)=\frac{\vartheta(\tau,2w)^2\vartheta(\tau,4w)}{\eta(\tau)}=q_3(\tau,2w)\,. \tag{196}$$

Introduce the three-torsion operators

$$\begin{aligned}(\mathcal{U}f)(\tau,w)&=f\left(\tau,w+\frac{1}{3}\right)\,,\\ (\mathcal{V}_m f)(\tau,w)&=\exp\left[2\pi i m\left(\frac{\tau}{9}+\frac{2w}{3}\right)\right]f\left(\tau,w+\frac{\tau}{3}\right)\,,\end{aligned} \tag{197}$$

and

$$\mathcal{P}_a=\frac{1}{3}\sum_{j=0}^{2}\omega^{-aj}\mathcal{U}^j\,,\qquad a=0,1,2\,.$$

A basis adapted simultaneously to the Heisenberg and modular matrices of Eqs. (170) and (171) is

$$\boldsymbol{Y}^{(1,m)}(\tau,w)=\begin{pmatrix}\mathcal{P}_0\mathcal{V}_m q_m\\ \mathcal{P}_1 q_m\\ -\mathcal{P}_2 q_m\end{pmatrix}\,,\qquad m=3,12\,. \tag{198}$$

We denote

$$F(\tau,w)\equiv\boldsymbol{Y}^{(1,3)}(\tau,w)\,,\qquad G(\tau,w)\equiv\boldsymbol{Y}^{(1,12)}(\tau,w)\,.$$