SO(3) Approach to Neutrino Mixing

Biswajit Karmakar
University of Silesia
Katowice, Poland

2509.25328, PLB 874 (2026) 140219 with Jaroslaw Duda & Janusz Gluza

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Neutrino parameters and the known unknowns

- Neutrinos are special!! It's flavor and mass eigenstates are related by :

$$|\nu_e, \nu_\mu, \nu_\tau\rangle^T_{\rm flavor} = U_{\alpha i}|\nu_1, \nu_2, \nu_3\rangle^T_{\rm mass}$$

- Pontecorvo-Maki-Nakagawa-Sakata parametrization: $\boxed{U_{\alpha i} = U_{\rm PMNS}}$

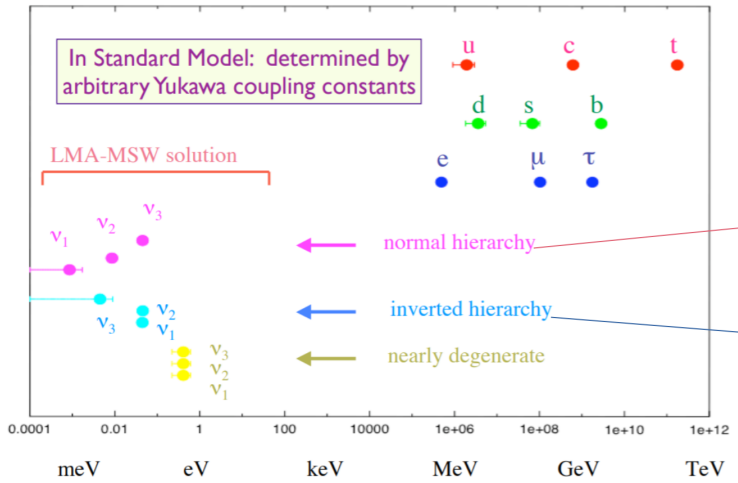
$$U_{\rm PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & s_{13}c_{23} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

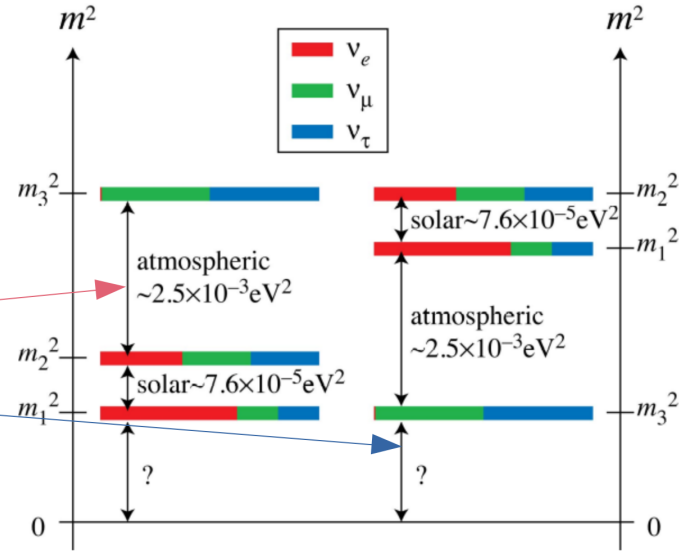
here $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. Figure : D. Meloni, 1709.02662

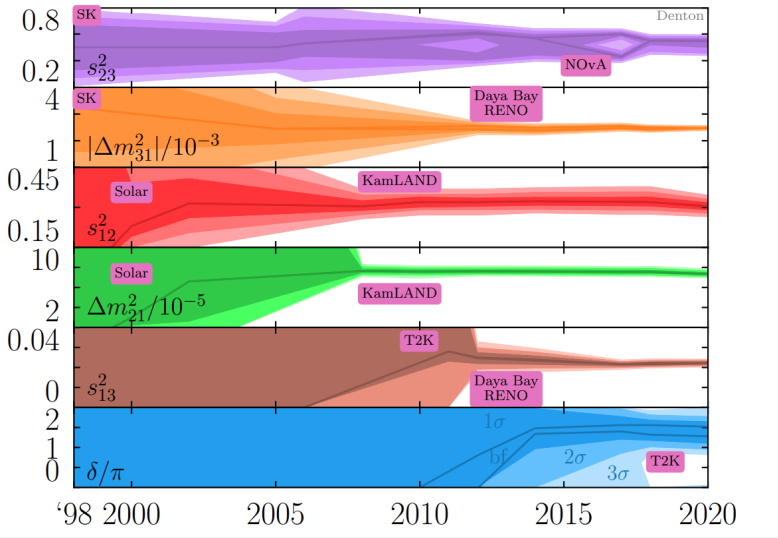
$$\|V_{CKM}\| = \quad \|U_{\rm PMNS}\| =$$



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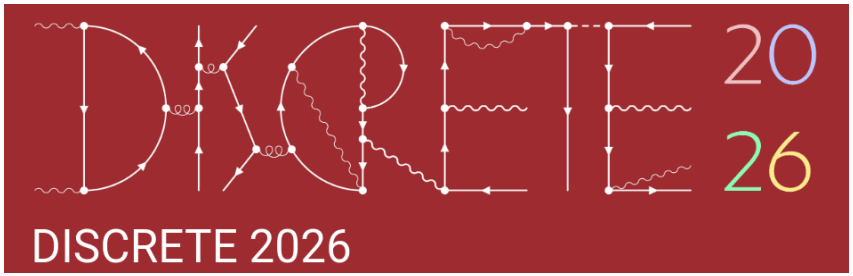






	Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 2.6$)	
	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.269 \to 0.343$	$0.304^{+0.012}_{-0.012}$	$0.269 \to 0.343$
$\theta_{12}/^\circ$	$33.44^{+0.77}_{-0.74}$	$31.27 \to 35.86$	$33.45^{+0.77}_{-0.74}$	$31.27 \to 35.87$
$\sin^2\theta_{23}$	$0.573^{+0.018}_{-0.023}$	$0.405 \to 0.620$	$0.578^{+0.017}_{-0.021}$	$0.410 \to 0.623$
$\theta_{23}/^\circ$	$49.2^{+1.0}_{-1.3}$	$39.5 \to 52.0$	$49.5^{+1.0}_{-1.2}$	$39.8 \to 52.1$
$\sin^2\theta_{13}$	$0.02220^{+0.00068}_{-0.00062}$	$0.02034 \to 0.02430$	$0.02238^{+0.00064}_{-0.00062}$	$0.02053 \to 0.02434$
$\theta_{13}/^\circ$	$8.57^{+0.13}_{-0.12}$	$8.20 \to 8.97$	$8.60^{+0.12}_{-0.12}$	$8.24 \to 8.98$
$\delta_{CP}/^\circ$	194^{+52}_{-25}	$105 \to 405$	287^{+27}_{-32}	$192 \to 361$
$\frac{\Delta m^2_{21}}{10^{-5}\ \text{eV}^2}$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$	$7.42^{+0.21}_{-0.20}$	$6.82 \to 8.04$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \text{eV}^2}$	$+2.515^{+0.028}_{-0.028}$	$+2.431 \to +2.599$	$-2.498^{+0.028}_{-0.029}$	$-2.584 \to -2.413$

Neutrino parameters and the known unknowns

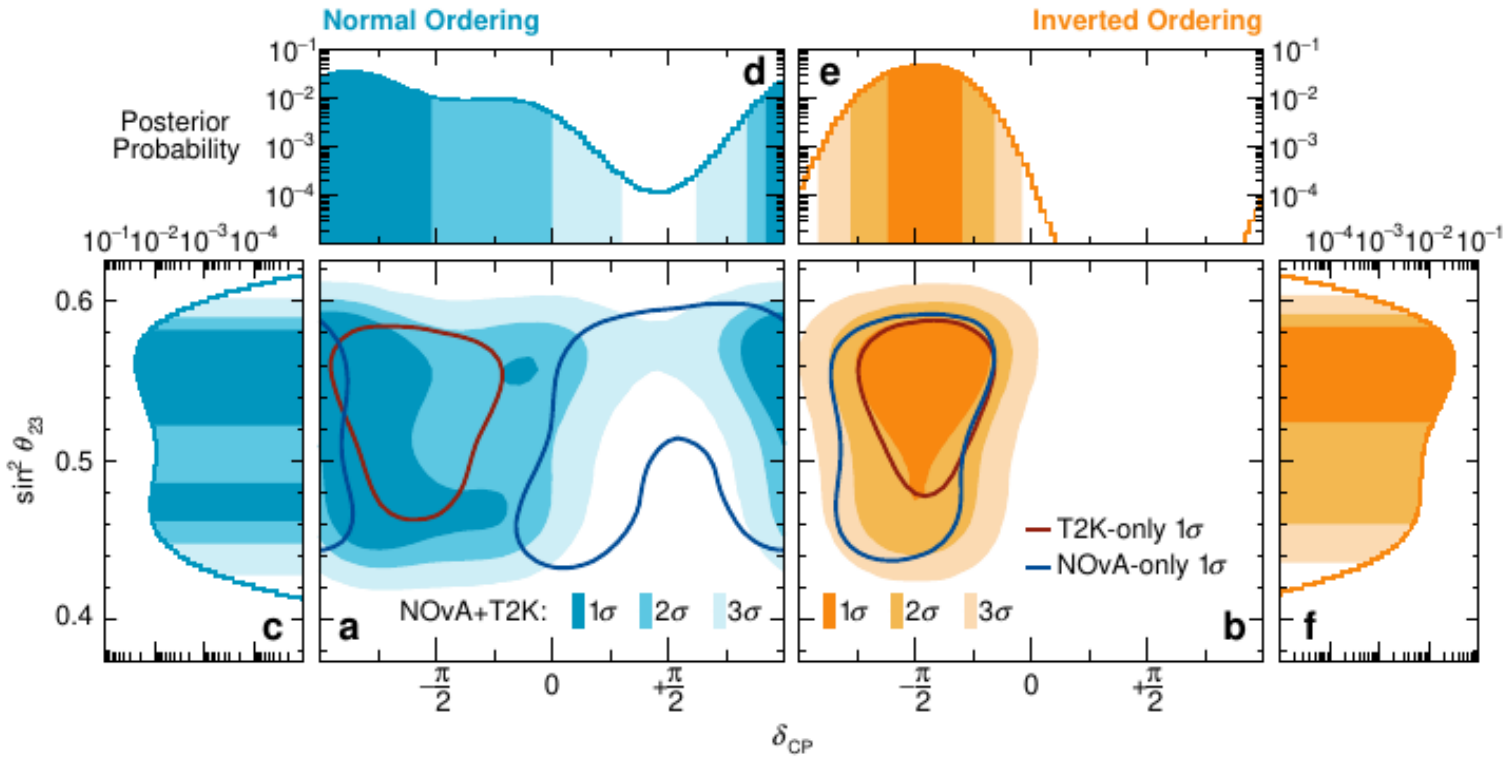


See talks

- Ana M. Teixeira
- Claudia Hagedorn
- Maria Concepcion Gonzalez-Garcia
- Thomas Schwetz-Mangold
- Tomas Nosek
- Pasquale Di Bari
- Antonio Marrone
- Ferruccio Feruglio
- Mu-Chun Chen
- Vit Vorobel
- Fedor Simkovic
- many more...

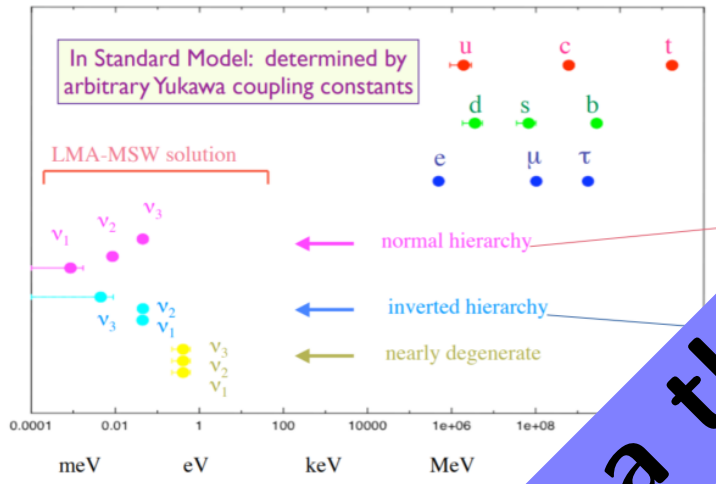
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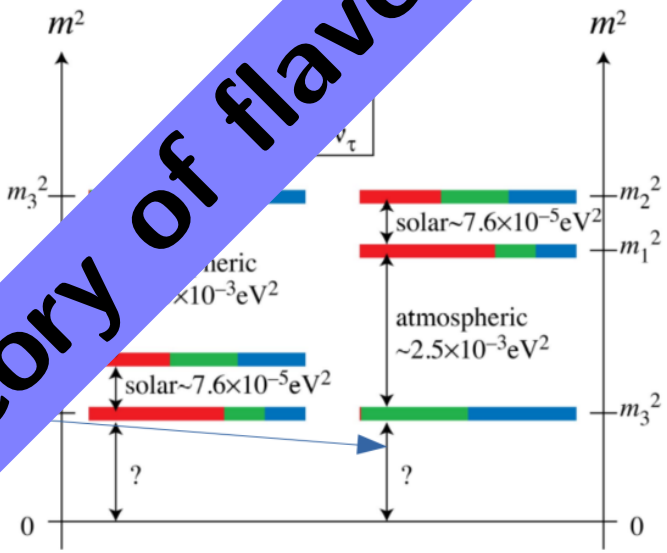
Joint neutrino oscillation analysis from the T2K and NOvA experiments; Nature 646, 818–824 (2025)



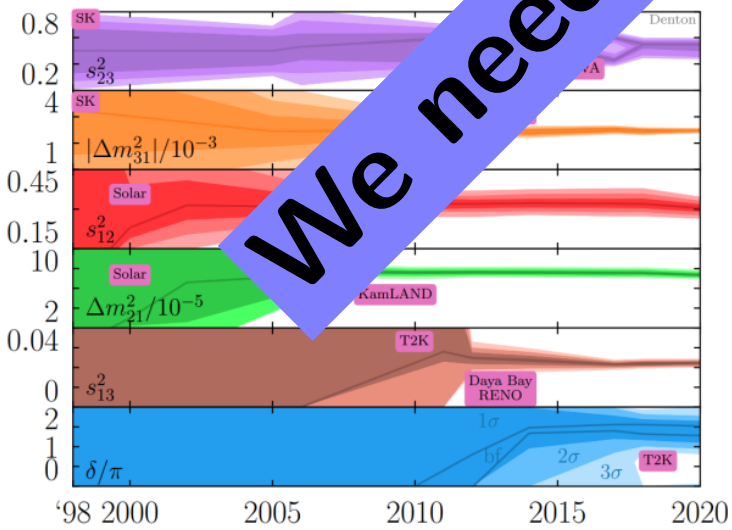
CP-conserving $\delta_{\mathrm{CP}} = 0/180^\circ$ is still possible!!

Neutrino parameters and the known unknowns





We need a theory of flavor!!



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Common wisdom to explain lepton flavor mixing

Anarchy

- Neutrino mixing anarchy is the hypothesis that the leptonic mixing matrix can be described as the result of a random draw from an unbiased distribution of unitary 3×3 matrices.
- Random analysis without imposing prior theories or symmetries on the mass and mixing matrices.
- This hypothesis does not make any correlation among the neutrino masses and mixing parameters

de Gouvea, Haba, Hall, Murayama : 9911341, 0009174, 1204.1249

Pheno Review: Progress in Particle and Nuclear Physics, 138 (2024) 104126

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Texture

- More specific studies with imposed mass or mixing textures for which models with underlying symmetries can be sought.
- It's an intermediate approach
- Some texture zeros of neutrino mass matrices can be eliminated.

Alejandro Ibarra, Graham Ross: Phys.Lett.B 2003

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Symmetry

- Theoretical studies where some explicit symmetries at the Yukawa Lagrangian level are assumed and corresponding extended particle sector is defined.
- The symmetry-based approach to explain the non-trivial mixing in the lepton sector known as family symmetry or horizontal symmetry

Review: Altarelli, Feruglio (Rev. Mod. Phys.) 1002.0211

Pheno Review: Progress in Particle and Nuclear Physics, 138 (2024) 104126

Symmetries, why? Theory Perspective

Lagrangian/neutrino mass mechanism $\otimes$ **Symmetry** $\Rightarrow$ **Observed Mixing**

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Example: $\mu - \tau$ permutation symmetry and tribimaximal (TBM) mixing

$$m_\nu = U_0^\star \mathrm{diag}(m_1, m_2, m_3) U_0^\dagger,$$

$$m_\nu = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix},$$

With $A + B = C + D$ this matrix yields TBM mixing pattern where $s_{12} = 1/\sqrt{3}$

- Compatible TBM Mixing Matrix :

$$|U_0| \simeq \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Neutrino Oscillation: as we measure it and $U_{\rm PMNS}$

- **Measured mixing matrix:**

$$|U_{\alpha i}| \sim \begin{pmatrix} 0.79 - 0.86 & 0.50 - 0.61 & 0.14 - 0.16 \\ 0.24 - 0.52 & 0.44 - 0.69 & 0.63 - 0.79 \\ 0.26 - 0.52 & 0.47 - 0.71 & 0.60 - 0.77 \end{pmatrix}$$

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Parametrization: non-exhaustive list

- Schechter, Valle PRD 80; Gerard, Xing PLB 12; Fritzsch, Xing PLB 01;
- Symmetric Parameterization Schechter,Valle PRD 81; Rodejohann,Valle PRD 11
- Expansion parameter: Rodejohann PRD 04; Li Ma PRD 05;
- Exponential: Zhukovsky, Borisov EPJC 16; Davydova, Zhukovsky PAN 19;
- many more....

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- **PDG/PMNS (Pontecorvo-Maki-Nakagawa-Sakata) parametrization:**

$$U_{\rm PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. In this convention, we have **three rotation angles and one phase**.

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where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. In this convention, we have **three rotation angles and one phase**.

- Elements of $U_{\alpha i}$ are not necessarily angles, however, such 'angle' parametrization has its advantages.

Neutrino Oscillation: as we measure it and $U_{\rm PMNS}$

- **PMNS parametrization:**

$$U_{\rm PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  - Such structure is inherited from the quark sector.

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  - Such structure is inherited from the quark sector.
  - Position of the imaginary part in the matrix is up to parametrization.
  - Encapsulates historic smallness of U_{e3}
  - **Warning!! We are not fitting any fundamental objects, it's just elements of some parametrization that we have chosen.**

Neutrino Oscillation

- Order dependent mixing matrix

$$U_{\rm PMNS} = R_x(\theta_{23})R_y(\theta_{13},\delta_{\rm CP})R_z(\theta_{12}) = U$$
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix}$$

- What we actually measure: Oscillation Probabilities

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathrm{Re}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right) + 2\sum_{i>j}\mathrm{Im}\left(U_{\alpha i}U^*_{\beta i}U^*_{\alpha j}U_{\beta j}\right)\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)$$

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- The expected number of signal events N_i in an energy bin i at a far detector:

$$N_i = \tau\, N_t \int_{E_i}^{E_{i+1}} dE_{\rm rec} \int_0^\infty dE_\nu\ \Phi(E_\nu)\cdot P_{\alpha\to\beta}(E_\nu, L; \boldsymbol{\theta}, \Delta m^2_{ij})\cdot\sigma_\beta(E_\nu)\cdot R(E_\nu, E_{\rm rec})\cdot\epsilon(E_{\rm rec})$$

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τ, N_t: Exposure time and total number of target nuclei in the detector.
$\Phi(E_\nu)$: Unoscillated incoming neutrino flux of flavor α.
$P_{\alpha\to\beta}$: Osc prob containing U or $U_{\rm PMNS}$ parameters.
$\sigma_\beta(E_\nu)$: **Neutrino-nucleus interaction cross-section for the detected flavor** β.
$R(E_\nu, E_{\rm rec})$: Detector energy resolution function mapping true energy E_ν to reconstructed energy $E_{\rm rec}$.
$\epsilon(E_{\rm rec})$: Event selection and reconstruction efficiency.

Neutrino Oscillation and U_{PMNS}

- Order-dependent mixing matrix :

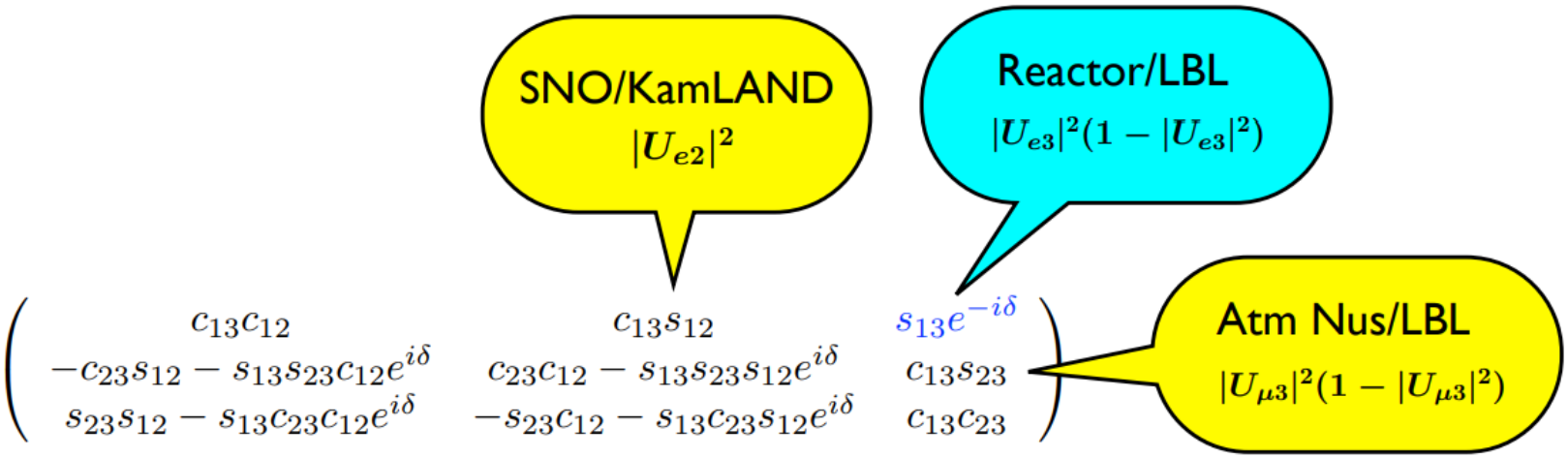
$$
\begin{aligned}
U_{PMNS} &= R_x(\theta_{23})R_y(\theta_{13},\delta_{\rm CP})R_z(\theta_{12})U_M(\alpha_1,\alpha_2) = U \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}
\begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\rm CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta\ {\rm CP}} & 0 & c_{13} \end{pmatrix}
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\end{aligned}
$$

'Secrets' of the Standard Parametrization

- The mass hierarchy ($\Delta m^2_{21} \ll |\Delta m^2_{31}|$, one-mass-scale dominance approximation) and the smallness of θ_{13}
- Placing $R_x(\theta_{23})$ (atmospheric) on the left and $R_z(\theta_{12})$ (solar) on the right $\Rightarrow$ extreme physical limit $\Rightarrow$ clean 2×2 framework.

Neutrino Oscillation and U_{PMNS}

- Elements of U_{PMNS}



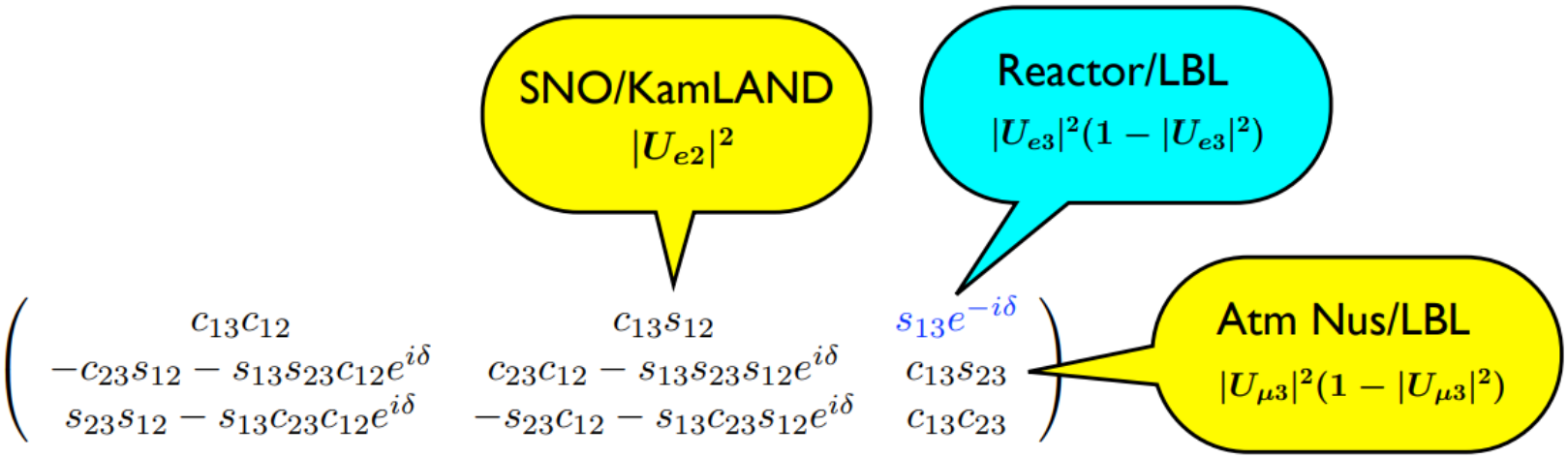
- Experimentally, with $\delta_{\rm CP} = 180^\circ$

$$U \to V_{\rm osc} = \begin{pmatrix} 0.817 \leftrightarrow 0.831 & 0.535 \leftrightarrow 0.558 & -0.150 \leftrightarrow -0.146 \\ -0.313 \leftrightarrow -0.273 & 0.615 \leftrightarrow 0.657 & 0.686 \leftrightarrow 0.738 \\ 0.471 \leftrightarrow 0.498 & -0.569 \leftrightarrow -0.519 & 0.658 \leftrightarrow 0.713 \end{pmatrix}$$

- Even with other CP conserving scenario, $\delta_{\rm CP} = 0^\circ$, measured values of $\theta_{23}, \theta_{13}, \theta_{12}$ remains same. $U_{\rm PMNS}$ does not distinguish the CP-conserving scenarios.

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- For CP Violation, fundamental quantity to describe the amount of CPV in lepton sector: $\Rightarrow \mathcal{J} \equiv {\rm Im}[U_{e1} U^*_{e2} U^*_{\mu 1} U_{\mu 2}]$; phase-convention-independent, representation-invariant, basis-independent measure

More on $U_{\rm PMNS}$

- The standard parameterization of the U matrix is one of six possible products of three-dimensional rotations for 3 flavors
- We adopted : Small-angle approximations $\Rightarrow$ usual oscillation paradigm

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U_{PMNS} &= R_x(\theta_{23},0)R_y(\theta_{13},\delta_{\rm CP})R_z(\theta_{12},0) \\
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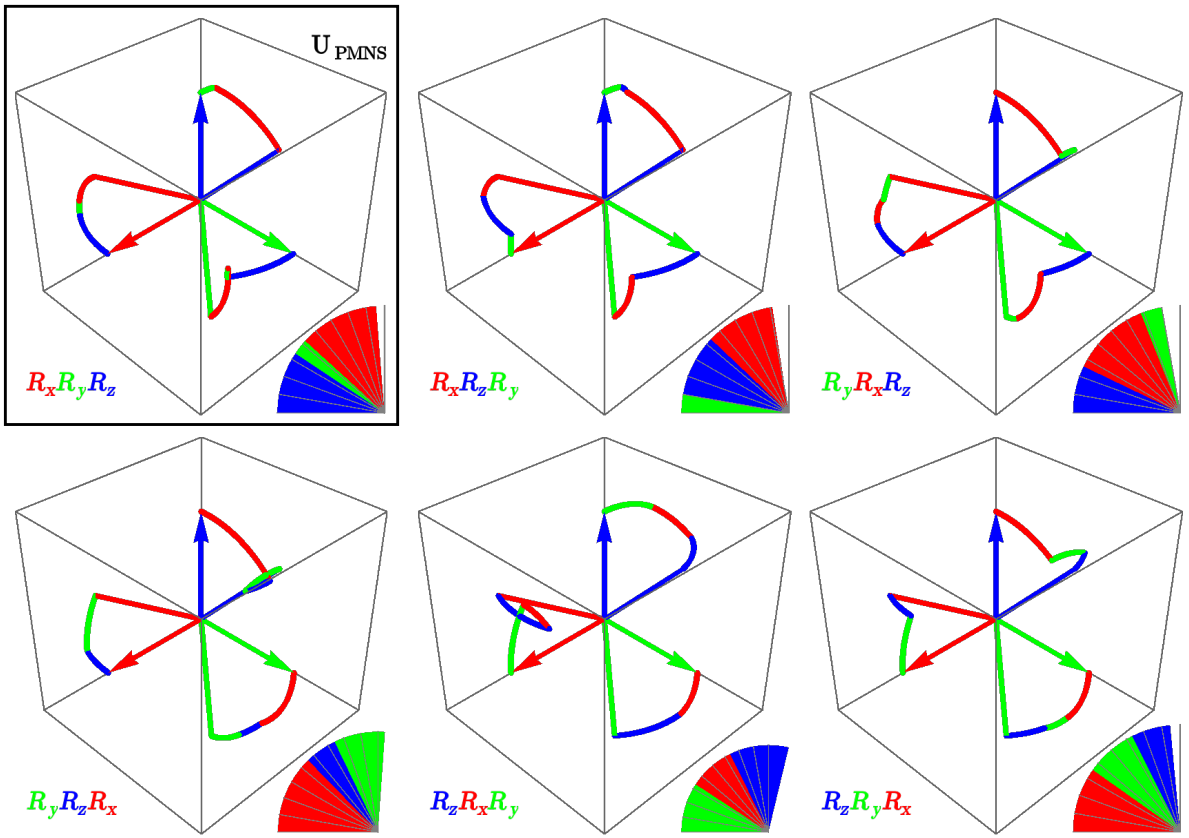
- Other possibilities:

$$R_xR_zR_y, \quad R_yR_xR_z, \quad R_yR_zR_x, \quad R_zR_xR_y, \quad R_zR_yR_x$$

"Who ordered that?"-Isidor Rabi (90 years ago after muon discovery)

More on $U_{\rm PMNS}$

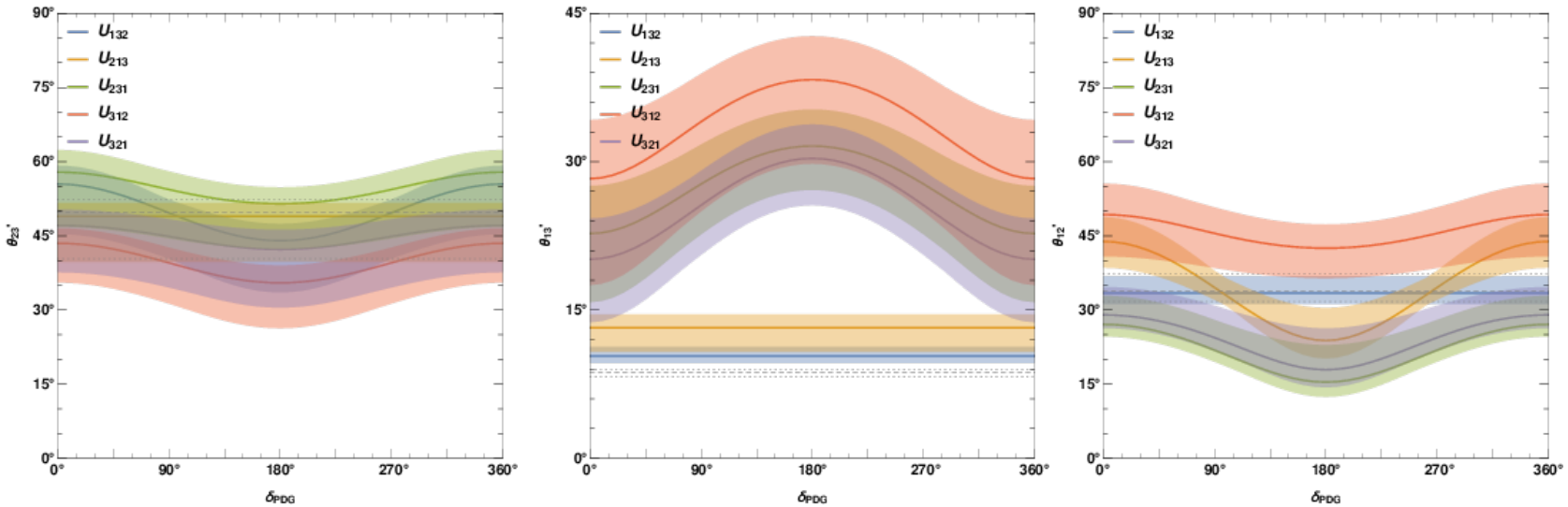
- Six possibilities : With $\delta_{\rm CP} = 180^\circ$



$$
\begin{aligned}
U &= R_x(43.3^\circ)R_y(8.56^\circ)R_z(33.68^\circ) = R_x(37.64^\circ)R_z(10.25^\circ)R_y(33.26^\circ) \\
&= R_y(42.70^\circ)R_x(11.69^\circ)R_z(25.70^\circ) = R_y(18.58^\circ)R_z(45.68^\circ)R_x(29.76^\circ) \\
&= R_z(33.17^\circ)R_x(39.63^\circ)R_y(30.71^\circ) = R_z(35.36^\circ)R_y(28.06^\circ)R_x(21.17^\circ).
\end{aligned}
$$

More on U

- Six possibilities : with $\delta_{\rm CP}$ varying



Mixing angles in each parameterization as a function of $\delta_{\rm CP}$

Denton, Pestes : 2006.09384

More on U_{PMNS}

- After decades of observation :

$$V_{\rm osc} = \begin{pmatrix} 0.817 \leftrightarrow 0.831 & 0.535 \leftrightarrow 0.558 & -0.150 \leftrightarrow -0.146 \\ -0.313 \leftrightarrow -0.273 & 0.615 \leftrightarrow 0.657 & 0.686 \leftrightarrow 0.738 \\ 0.471 \leftrightarrow 0.498 & -0.569 \leftrightarrow -0.519 & 0.658 \leftrightarrow 0.713 \end{pmatrix}$$

$$\boxed{|(V_{\rm osc})_{e1}|^2 > |(V_{\rm osc})_{\mu 1}|^2 > |(V_{\rm osc})_{\tau 1}|^2}$$

- Permutations of Euler matrices are qualitatively equivalent
- Mixing angles are parameters of the U matrix, not physical observables, fitted experimentally
- There is nothing special in choosing U_{PMNS} as far as neutrino mixing parametrization is concerned
- Current choice of U_{PMNS} is order dependent
- Specific representation matters as theoretical considerations and mass/mixing models are based on this choice.

Neutrino Mixing

Can ν mixing originate directly from any symmetry principle?

Lagrangian/neutrino mass mechanism $\otimes$ [Symmetry $\Rightarrow$ Observed Mixing]

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Special Orthogonal group in 3 dimensions, denoted as SO(3)

SO(3) Group Theory

- The continuous SO(3) family symmetry is one of the most widely used Lie groups in mathematics and physics. Mathematically, SO(3) is the set of all (3×3) real matrices R that satisfy: $\boxed{R^T R = \mathbf{I} \text{ and } \det(\mathrm{R})=+1}$.

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- Fundamental three-dimensional representaion of the standard antisymmetric SO(3) generators :

$$G_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, G_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, G_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $G_i^T = -G_i$ and the commutator of the generators can be written as

$$[G_i, G_j] = \epsilon_{ijk} G_k, \quad i, j, k = x, y, z.$$

SO(3) and Euler rotaions

- Euler rotations are elements of SO(3):
- The matrix $R_x(\theta_{23}), R_y(\theta_{13}, \delta_{\rm CP} = 180^\circ), R_z(\theta_{12})$ can be obtained via SO(3) generators G_x, G_y, G_z

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U &= e^{-\theta_{23} G_x} e^{-\theta_{13} G_y} e^{-\theta_{12} G_z}, \\
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- Other combinations, such as $e^{-\theta_{23} G_x} e^{-\theta_{12} G_z} e^{-\theta_{13} G_y}$ or $e^{-\theta_{12} G_z} e^{-\theta_{23} G_x} e^{-\theta_{13} G_y}$ will imply different values of the mixing angles, not reflecting nearly maximal mixings in U_{PMNS}

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U_{SO3}
U_{PMNS}

SO(3) order independent parameterization

- How to avoid the order dependence for the multiplication of group matrices ?
- We propose, an *order independent parameterization* of rotations in a plane perpendicular to $\vec{\Theta} = (\theta_x, \theta_y, \theta_z)$ through the matrix exponent.
- Using the three SO(3) generators, and the three mixing angles $(\theta_x, \theta_y, \theta_z)$:

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- Using the three SO(3) generators, and the three mixing angles $(\theta_x, \theta_y, \theta_z)$:

$$U_{\rm SO3} = \exp(-\vec{\Theta}.\vec{G}) = \exp(-\theta_x G_x - \theta_y G_y - \theta_z G_z) = \exp\begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\theta_x^2+\cos\theta\left(\theta_y^2+\theta_z^2\right)}{\theta^2} & \frac{\theta\theta_z\sin\theta-\theta_x\theta_y(\cos\theta-1)}{\theta^2} & -\frac{\theta_x\theta_z(\cos\theta-1)+\theta\theta_y\sin\theta}{\theta^2} \\ -\frac{\theta_x\theta_y(\cos\theta-1)+\theta\theta_z\sin\theta}{\theta^2} & \frac{\cos\theta\left(\theta_x^2+\theta_z^2\right)+\theta_y^2}{\theta^2} & \frac{\theta\theta_x\sin\theta-\theta_y\theta_z(\cos\theta-1)}{\theta^2} \\ \frac{\theta\theta_y\sin\theta-\theta_x\theta_z(\cos\theta-1)}{\theta^2} & -\frac{\theta\theta_x\sin\theta+\theta_y\theta_z(\cos\theta-1)}{\theta^2} & \frac{\cos\theta\left(\theta_x^2+\theta_y^2\right)+\theta_z^2}{\theta^2} \end{pmatrix}.$$

where $\theta = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$.

$U_{\rm SO3}$ is also orthogonal and unimodular $\Rightarrow$ Rotation

SO(3) order independent parameterization

Warning !!

- The exponential form $U_{\rm SO3} = \exp(-\vec{\Theta}.\vec{G}) = \exp(-\theta_x G_x - \theta_y G_y - \theta_z G_z)$, is in general basis-dependent.
- Under a basis transformation, the magnitude of θ_x, θ_y and θ_y will change.

Good news !!

- The rotation itself, however, will remain the same, and all such forms are related by similarity transformation.
- Since similarity transformations preserve intrinsic quantities such as the rotation angle, the physical content of the transformation remains unchanged.
- Here we work with the fundamental three-dimensional representaion of the standard antisymmetric SO(3) generators (and their commutator relations)

SO(3) order independent parameterization

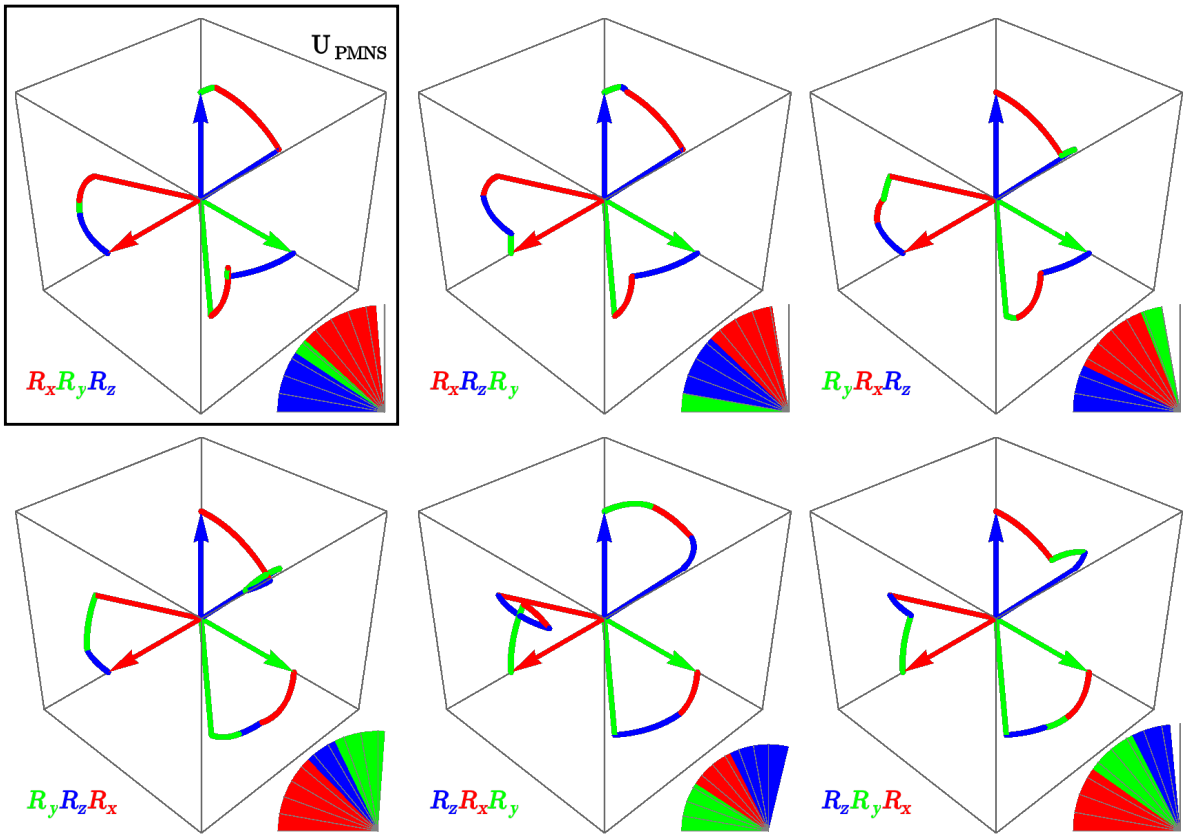
- $U_{\rm SO3}$ explains $V_{\rm osc}$, with $\delta_{\rm CP} = 180^\circ$

$$\theta_x = 43.8^\circ \pm 2.1^\circ, \quad \theta_y = 21.7^\circ \pm 0.6^\circ, \quad \theta_z = 28.3^\circ \pm 0.8^\circ$$

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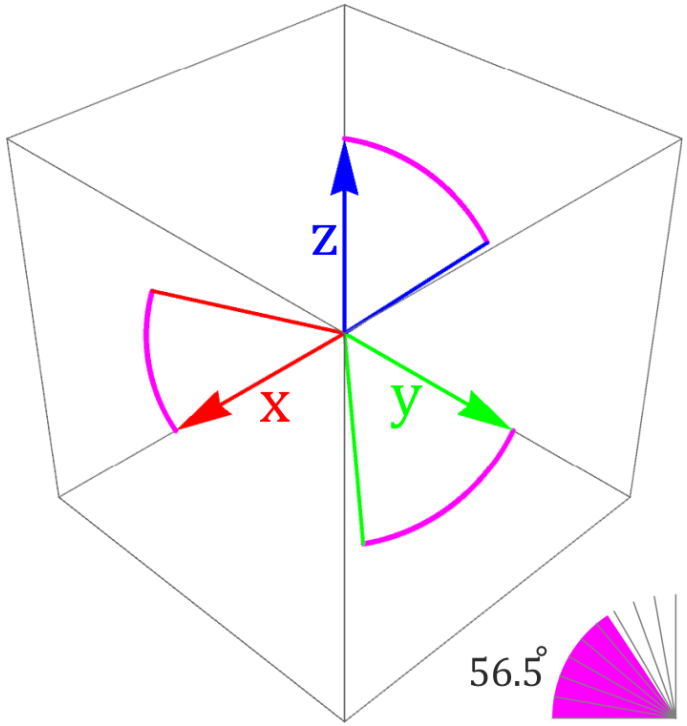
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- $\delta_{\rm CP} = 0^\circ$

$$\boxed{\theta_x = 46.5^\circ \pm 2.4^\circ, \quad \theta_y = 5.4^\circ \pm 0.8^\circ, \quad \theta_z = 35.1^\circ \pm 0.7^\circ}$$

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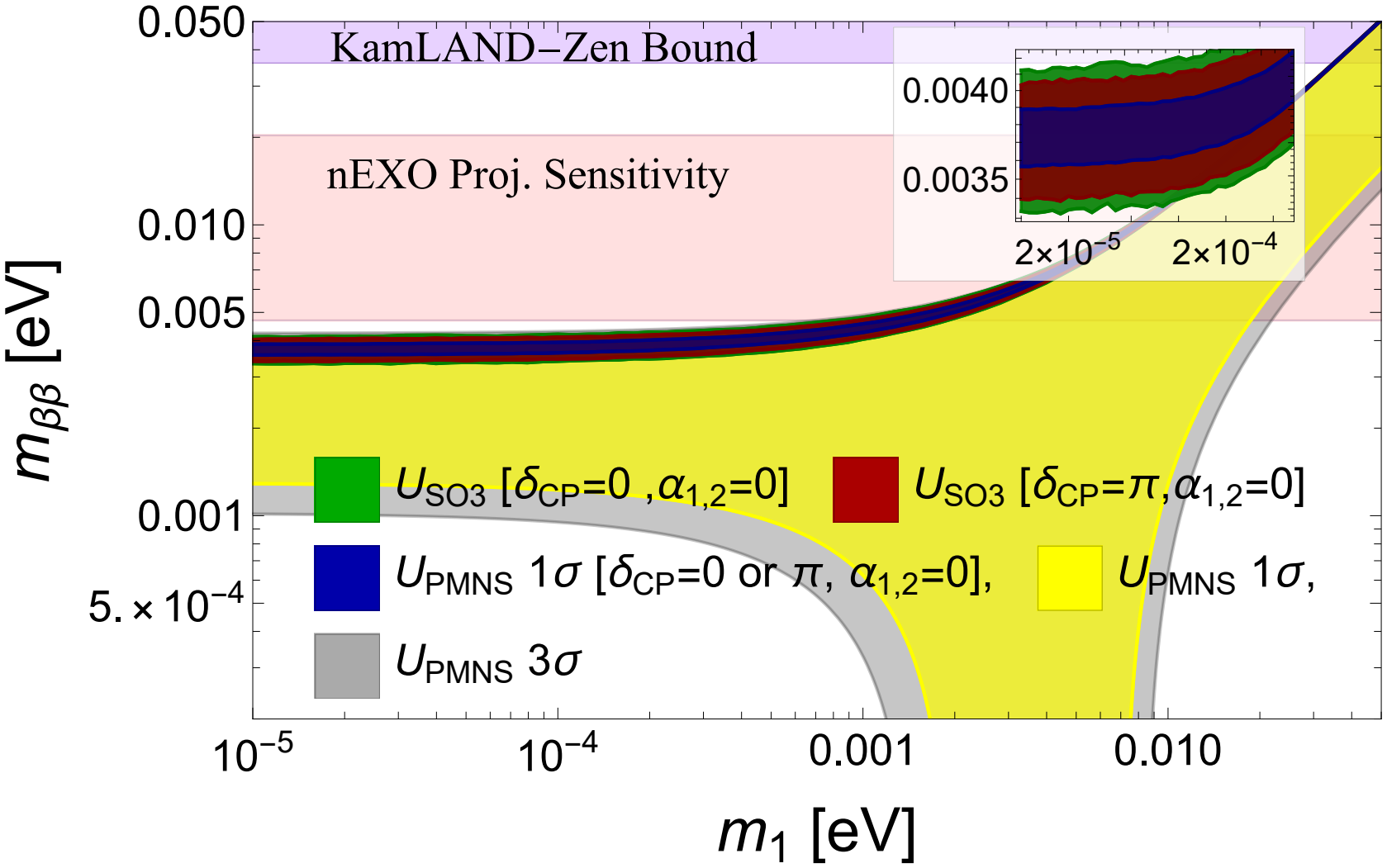
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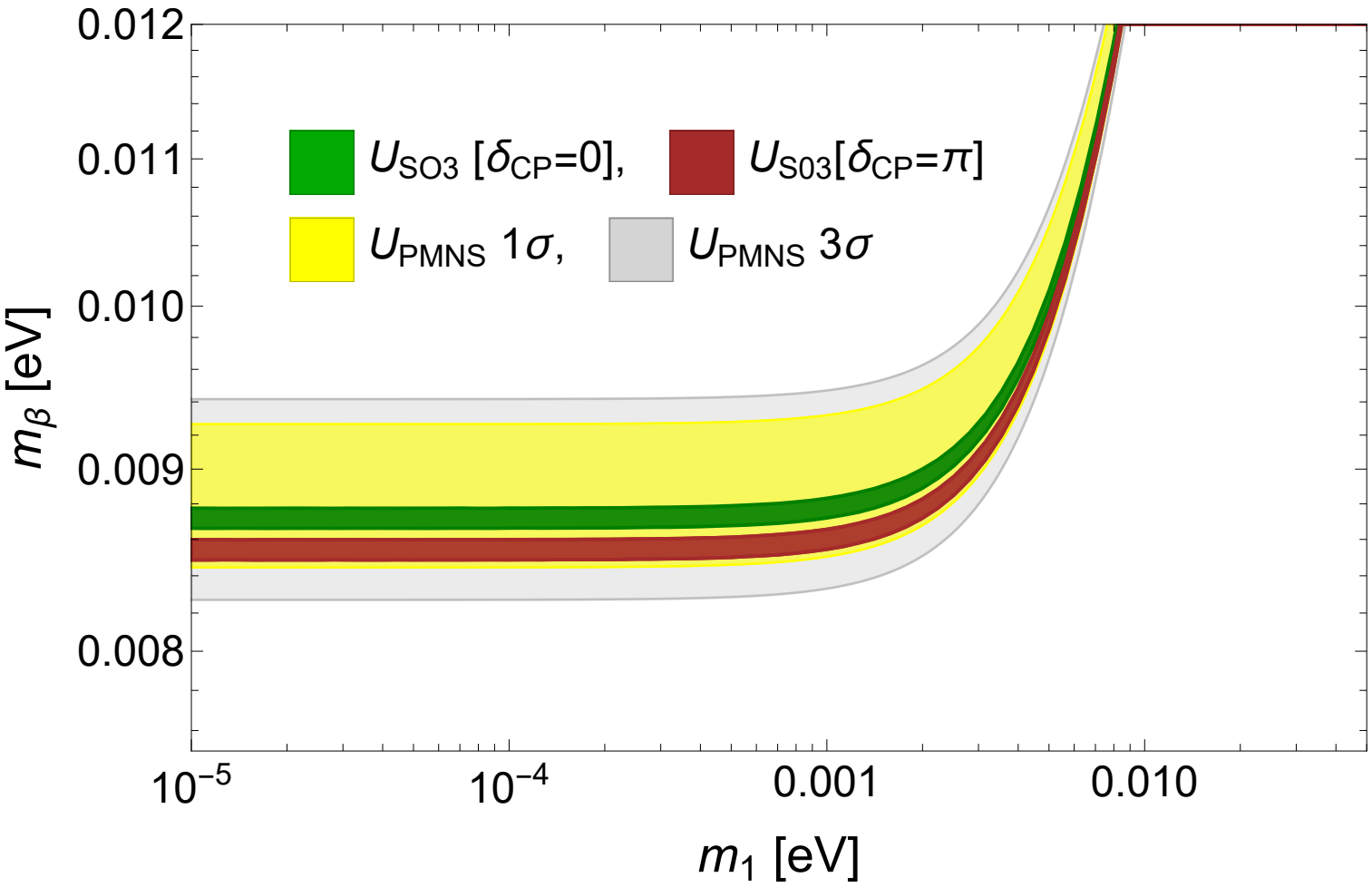
What do these results mean?

SO(3) parameterization: phenomenology I



$m_{\beta\beta} = \left| \sum_i m_i U_{ei}^2 \right|$, for NO with vanishing Majorana phases

SO(3) parameterization: phenomenology I



$$m_\beta^2 = \frac{\sum_i m_i^2 |U_{ei}|^2}{\sum_i |U_{ei}|^2} = \sum_i m_i^2 |U_{ei}|^2$$

SO(3) Parametrization : Outlook

- We propose a parameterization of the 3-dimensional U matrix based on the SO(3) group, which corresponds to one general rotation in 3-dimensional space.

- To harmonize the signs of elements of the SO(3) generators and related commutator relation with the neutrino mixing matrix that follow, we infer $\delta_{\rm CP} = 0/180^\circ$

- Yields near maximal mixing for $\delta_{\rm CP} = 0^\circ$, whereas relatively large, 'democratic' values of the mixing angles $\theta_x, \theta_y, \theta_z$ for $\delta_{\rm CP} = 180^\circ$

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- Yields near maximal mixing for $\delta_{\rm CP} = 0^\circ$, whereas relatively large, 'democratic' values of the mixing angles $\theta_x, \theta_y, \theta_z$ for $\delta_{\rm CP} = 180^\circ$

SO(3) Parametrization : Outlook

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Thank you!!

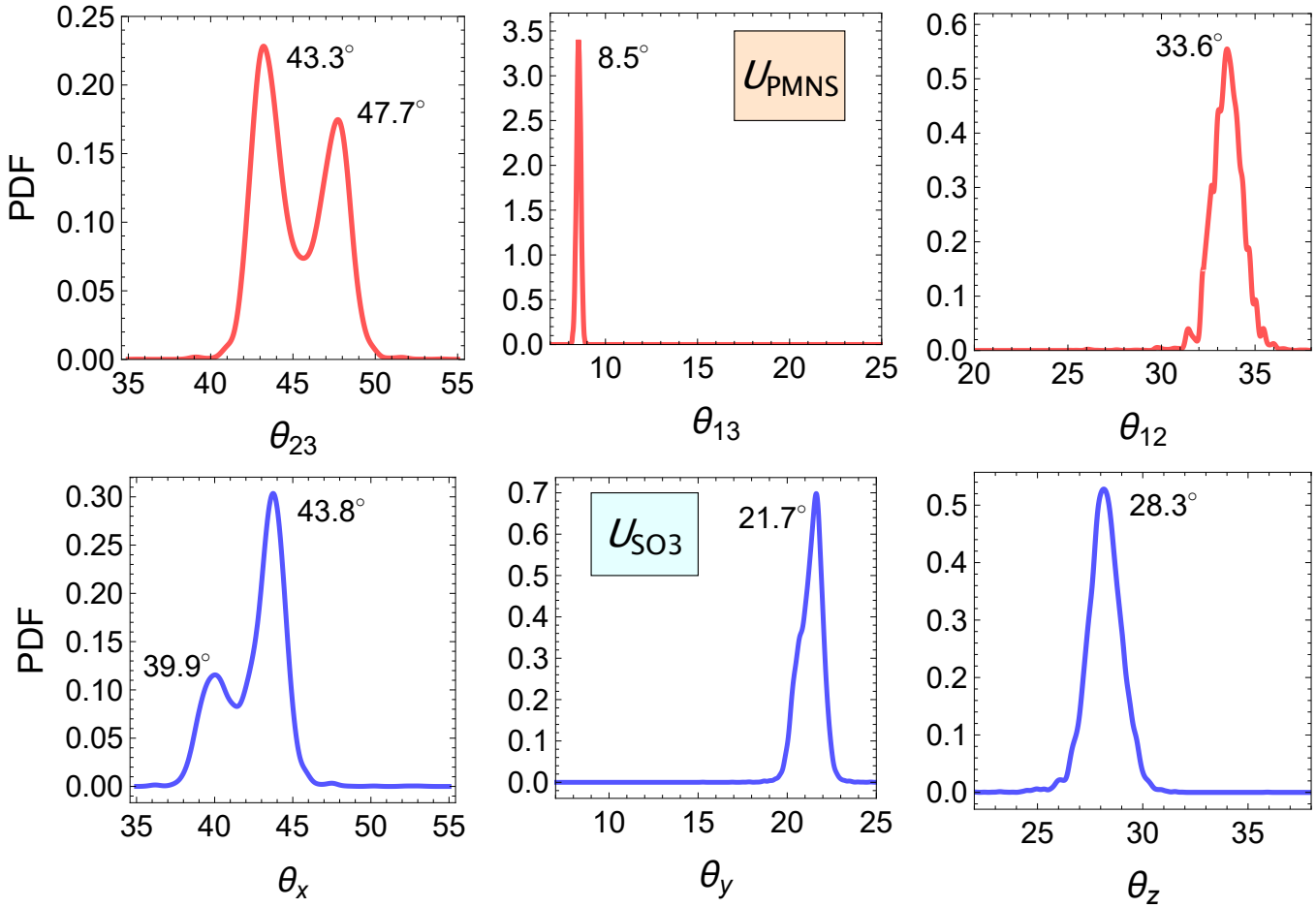
Back-up slides

Neutrino Oscillation and U_{PMNS}

Tuning the experimental baseline scale $\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E}$, specific frequencies decouple:

Experiment	Physical Limit / Regime	Structural Reduction	Parameters
Atmospheric	Long base-line $\Delta m^2_{21} \to 0$ $(\Delta_{31} \sim \pi/2, \Delta_{21} \to 0)$ ν_1, ν_2 mass-degenerate	$P_{\mu\mu} \approx$ $1 - \sin^2(2\theta_{23}) \cos^4(\theta_{13}) \sin^2\left(\frac{\Delta m^2_{31} L}{4E}\right)$ $R_z(\theta_{12})$ physically drops out $\theta_{13} \to 0 \implies \nu_e$ decouples	$\theta_{23}, \lvert\Delta m^2_{31}\rvert$
Solar	Very-long baseline $\Delta m^2_{31} \to \infty$ $(\Delta_{21} \sim \frac{\pi}{2}, \Delta_{31} >> 1)$ Rapid atmospheric oscillations average out $(\sin^2 \Delta_{31} \to \frac{1}{2})$; ν_3 kinematically decouples	$P_{ee} \approx \cos^4(\theta_{13}) \Big[1 - \sin^2(2\theta_{12}) \times$ $\sin^2\left(\frac{\Delta m^2_{21} L}{4E}\right)\Big] + \sin^4(\theta_{13})$ $R_Y(\theta_{13}) \to$ const. shift 2×2 solar block isolates	$\theta_{12}, \Delta m^2_{21}$

SO(3) order independent parameterization



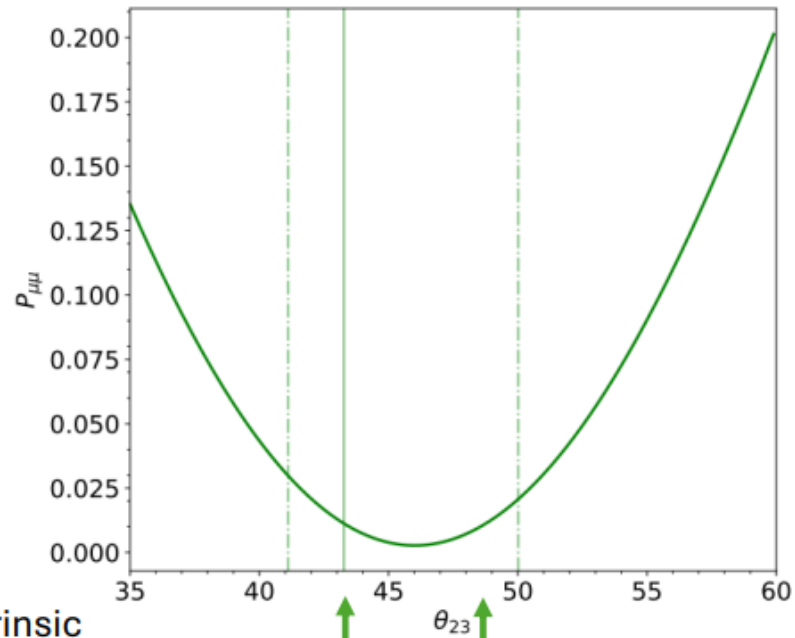
NuFIT χ^2 → probability distributions of mixing angles → Monte Carlo sampling → PMNS matrices → nearest SO(3) matrices → distributions of new angles

$$\theta_x = 43.8^\circ \pm 2.1^\circ, \quad \theta_y = 21.7^\circ \pm 0.6^\circ, \quad \theta_z = 28.3^\circ \pm 0.8^\circ$$

SO(3) parameterization: phenomenology II (preliminary)

Disappearance probability for DUNE (with Monojit Ghosh and Elena Abad Diaz)

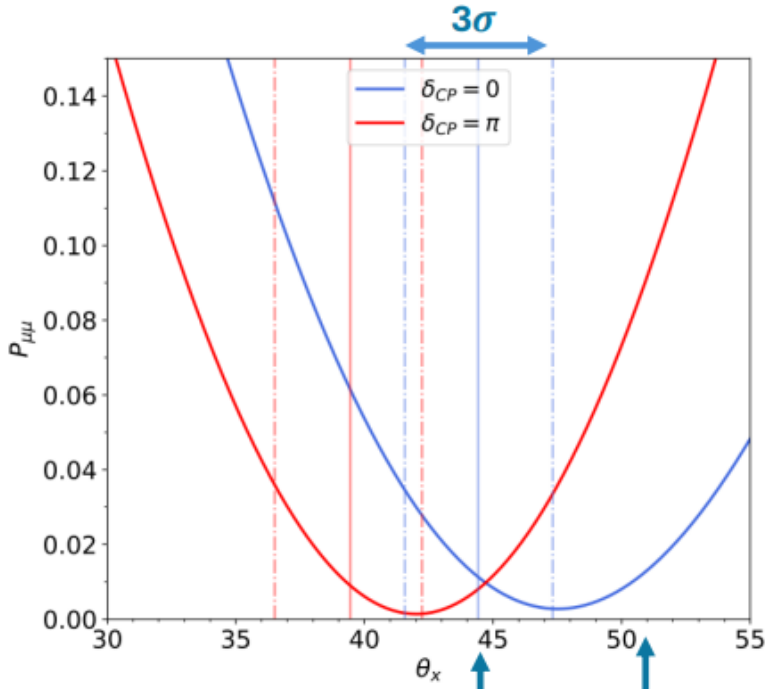
$$P_{\mu\mu} = 1 - \sin^2 2\theta_{23} \sin^2 \Delta$$



intrinsic octant degeneracy

$P_{\mu\mu}(\theta_{23}) = P_{\mu\mu}(\pi/2 - \theta_{23})$

Both inside 3σ



Best-fit

Clone solution → outside 3σ

SO(3) parameterization: phenomenology II (preliminary)

Appearance probability for DUNE

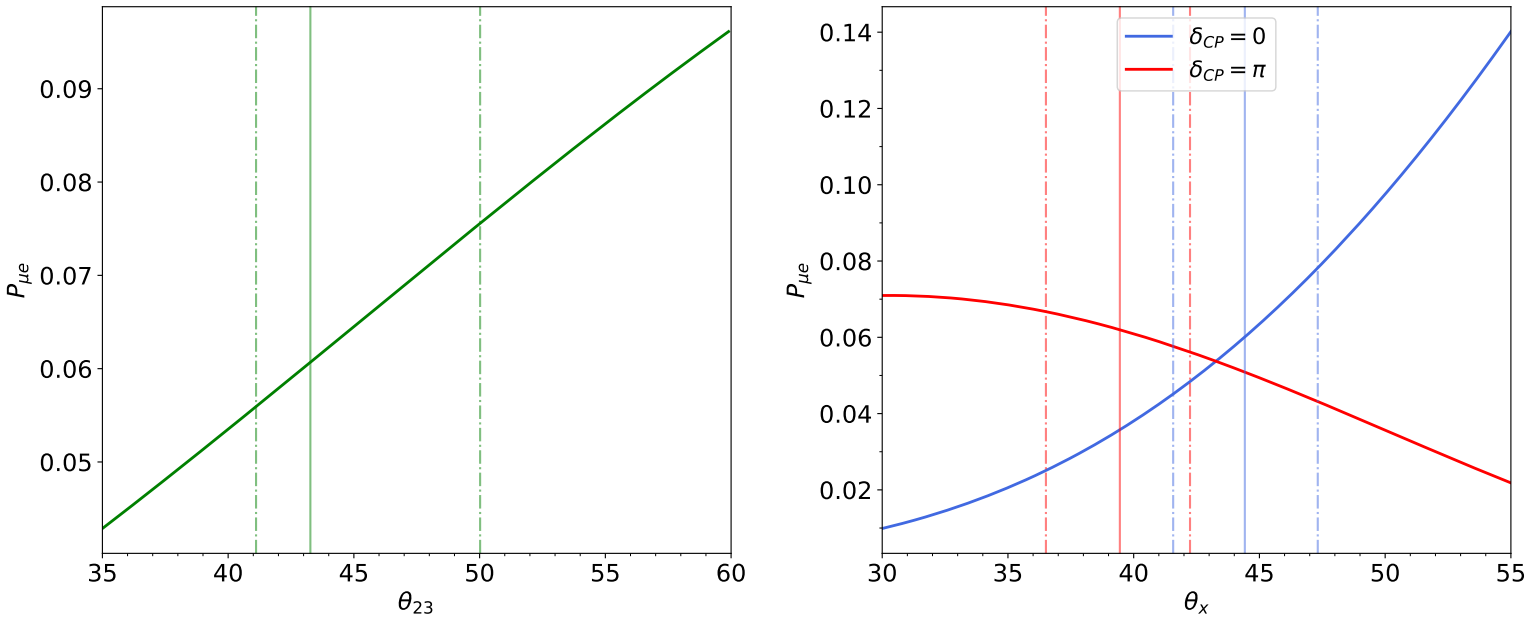


Figure: $P_{\mu e}$ vs θ_{23} (left) θ_x (right)

SO(3) parameterization: phenomenology II (preliminary)

Appearance probability for DUNE

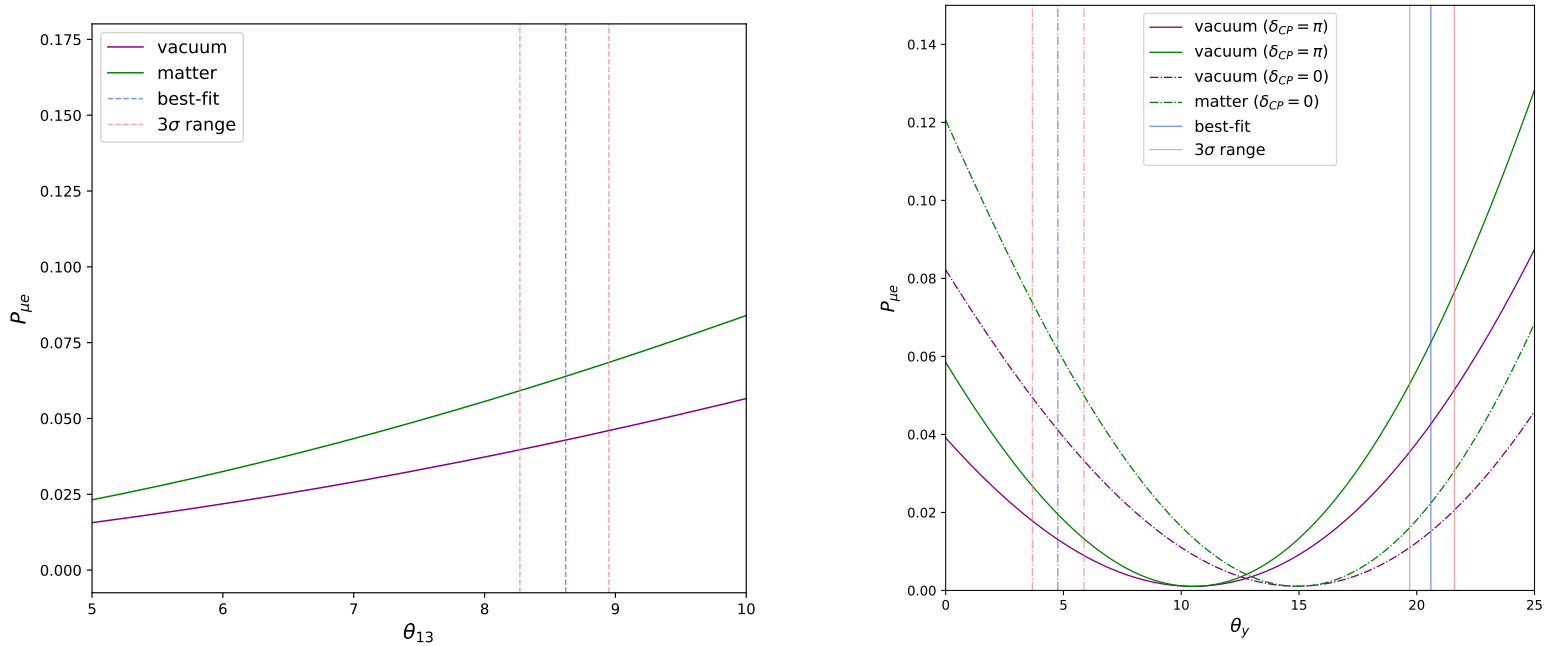


Figure: $P_{\mu e}$ vs θ_{13} (left) θ_y (right)

Detailed phenomenology to follow