UNIVERSITÀ DI PADOVA
Dipartimento
di Fisica e Astronomia
"Galileo Galilei"

INFN
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Modular Domain Walls and their Cosmological Implications

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10th Symposium on Prospects
in the Physics of Discrete Symmetries

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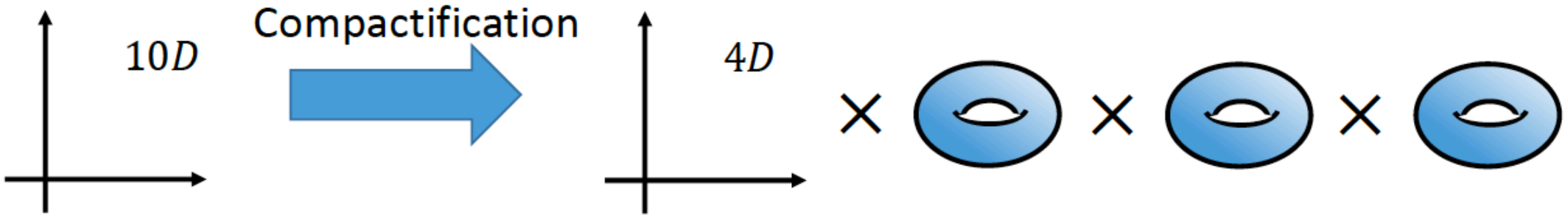
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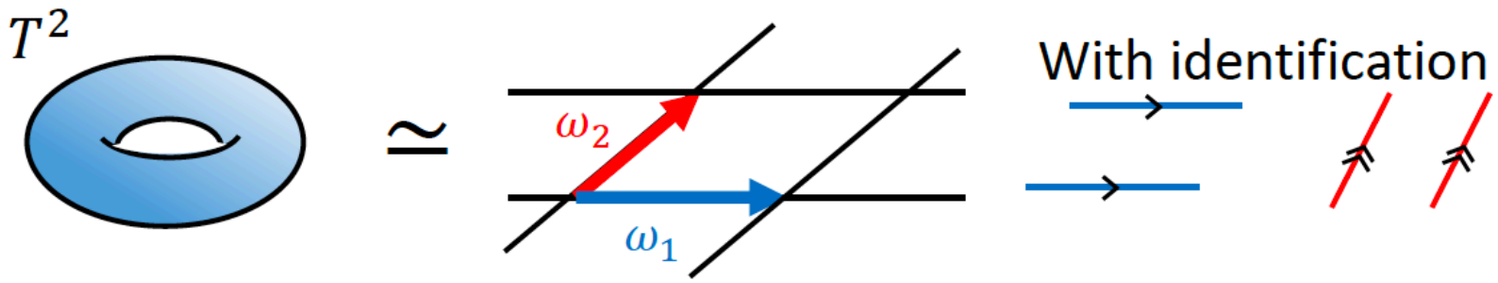
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Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





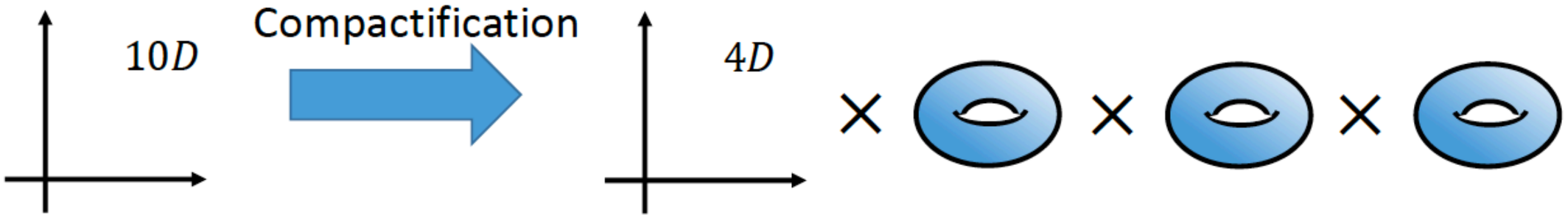
$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

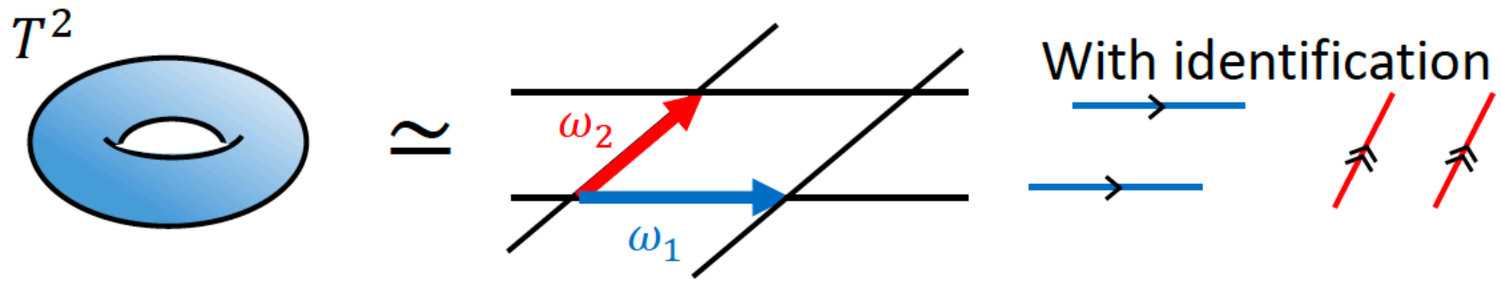
modulus

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$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

modulus

Lattice is invariant under basis transformations

$$\begin{cases} \omega_2 \to \omega_2' = a\omega_2 + b\omega_1 \\ \omega_1 \to \omega_1' = c\omega_2 + d\omega_1 \end{cases} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}$$

modular transformations

τ and τ' describe the same torus

Modular group

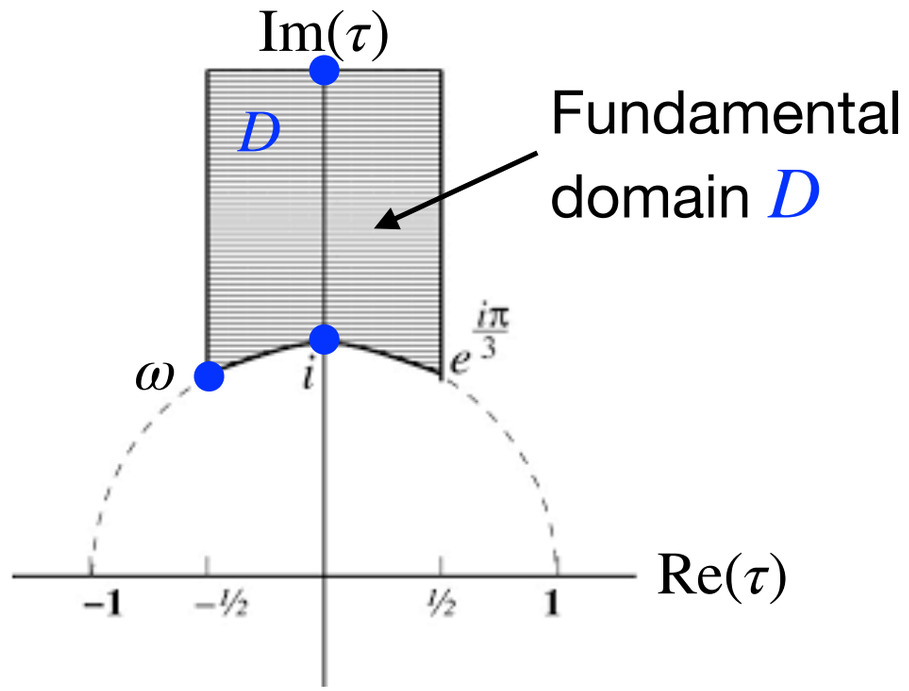
Homogeneous modular group

$$\Gamma = \langle S,\, T \mid S^4 = (ST)^3 = I,\, S^2T = TS^2 \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry



Modular group

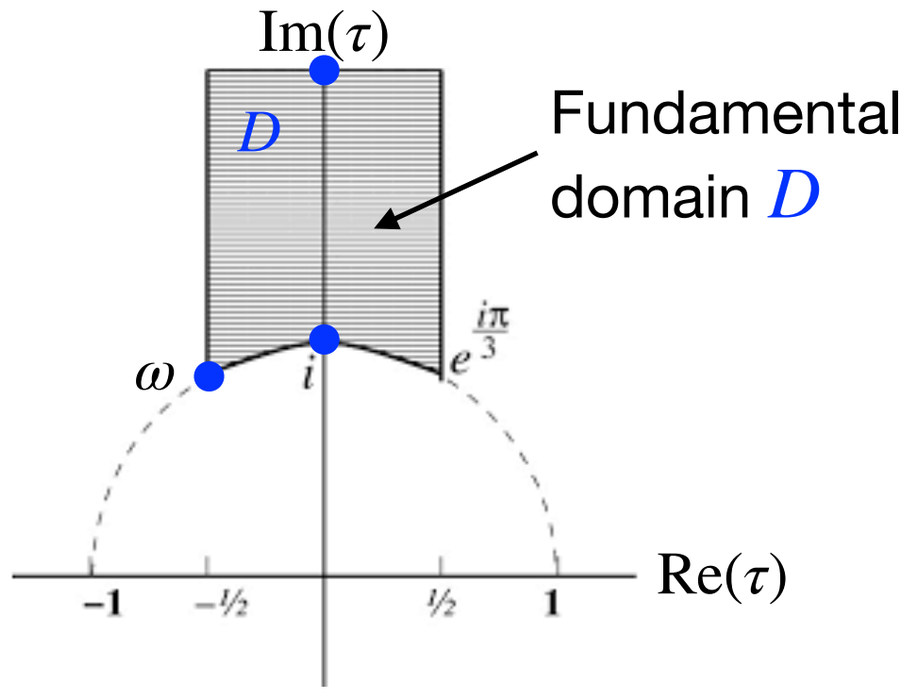
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duality — discrete shift symmetry



Fixed points

- $\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_4^S$
- $\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega + 1} = \omega \quad \Rightarrow \quad Z_3^{ST} \times Z_2^{S^2}$
- $\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z^T \times Z_2^{S^2}$

Modular-invariant SUSY theory

$\mathcal{N} = 1$ global SUSY action

Ferrara et al., PLB 1989

$$\mathscr{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K(\tau, \bar{\tau}) + \int \mathrm{d}^2\theta\; W(\tau) + \int \mathrm{d}^2\bar{\theta}\; \overline{W}(\bar{\tau})$$

Kähler potential K Superpotential W

In $\mathcal{N} = 1$ SUGRA, the action depends on $G(\tau, \bar{\tau}) = K(\tau, \bar{\tau})/\bar{M}_{\rm Pl}^2 + \ln |\, W(\tau)/\bar{M}_{\rm Pl}^3 \,|^2$

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Kähler potential and superpotential

Cvetic et al., NPB 1991

Modular-invariant function H

$$K(\tau,\bar{\tau}) = -\,3\Lambda_K^2 \ln(-i\tau + i\bar{\tau}) \quad \text{and} \quad W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^{2k_W}}\,, \quad k_W = 3\frac{\Lambda_K^2}{\bar{M}_{\rm Pl}^2}$$

Dedekind eta function η (weight 1/2)

Under modular transformations

$$K(\tau,\bar{\tau}) \to K(\tau,\bar{\tau}) + f(\tau) + \bar{f}(\bar{\tau}) \quad \text{with} \quad f(\tau) = 3\Lambda_K^2 \ln(c\tau + d)$$

$$W(\tau) \to (c\tau + d)^{-k_W} W(\tau)$$ Modular weight $-k_W$

$k_W = 0$ $(\bar{M}_{\rm Pl} \to \infty)$ in global SUSY, whereas $k_W = 3$ $(\Lambda_K = \bar{M}_{\rm Pl})$ in SUGRA

Flavour puzzle and strong CP problem

Coupling to matter

Feruglio, 1706.08749

$$W = \Psi_i^c \, Y_{ij}(\tau) \, \Psi_j \, \Phi$$

τ-dependent Yukawa couplings are modular forms

$$Y_{ij}(\tau) \to \left(c\tau + d\right)^{k_{Y_{ij}}} \, Y_{ij}(\tau)$$

Talks by Ferruccio Feruglio, Mu-Chun Chen, Salvador Centelles Chuliá

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Alternative solution to the strong CP problem

Feruglio, Strumia, **AT**, JHEP 2023

(modular invariance + spontaneous CP violation)

$$\bar{\theta} = \theta_{\rm QCD} + \arg\det\left(Y^u Y^d\right)$$

$$Y^q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_6(\tau) \\ 1 & E_6(\tau) & E_4(\tau)^3 + E_6(\tau)^2 \end{pmatrix}$$ s.t. $\det\left(Y^u Y^d\right)$ is real while Y^q are complex

Modular-invariant function H

The most general form without singularities
in the fundamental domain

Rademacher, Zuckerman, AoM, 1938

Integers $m \geq 0,\ n \geq 0$

$$H(\tau) = (J(\tau) - 1)^{m/2} J(\tau)^{n/3} P(J(\tau))$$

$J(\tau) = \dfrac{E_4(\tau)^3}{12^3\,\eta(\tau)^{24}} = \dfrac{E_6(\tau)^2}{12^3\,\eta(\tau)^{24}} + 1 = \dfrac{j(\tau)}{12^3}$ is Klein's j-invariant (modular invariant)

$E_4(\tau)$ and $E_6(\tau)$ are Eisenstein series (modular forms of weights 4 and 6)

$P(J(\tau)) = c_0 + c_1\,J(\tau) + c_2\,J(\tau)^2 + \ldots$ We will focus on $P = 1$

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Strictly speaking $H(\tau)$ is modular-invariant up to a τ-independent phase,
since $\eta(\tau)$ has a non-trivial multiplier system $v_\eta(S) = e^{-i\pi/4}$ and $v_\eta(T) = e^{i\pi/12}$

$$H(S\tau) = (-1)^m\,H(\tau)\,, \quad H(T\tau) = (-1)^m\,e^{-2\pi i\,n/3}\,H(\tau) \quad \Rightarrow \quad H((ST)\tau) = e^{-2\pi i\,n/3}\,H(\tau)$$

Thus $H(\tau)$ is modular-invariant only if m is even and n is a multiple of 3 (important for W)

Modular-invariant potentials

In SUGRA, the scalar potential

Cvetic et al., NPB 1991

$$V = e^{K/\bar{M}_{\rm Pl}^2}\left[K^{i\bar{j}} D_i W D_{\bar{j}} \overline{W} - 3\,|\,W\,|^2/\bar{M}_{\rm Pl}^2\right] \quad \text{with} \quad D_i = \partial_i + \partial_i K/\bar{M}_{\rm Pl}^2$$

With our $K(\tau,\bar{\tau})$ and $W(\tau)$

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{(2\,{\rm Im}\tau)^{k_W}\,|\,\eta(\tau)\,|^{4k_W}}\left[\left|iH'(\tau) + \frac{k_W}{2\pi}H(\tau)\hat{G}_2(\tau,\bar{\tau})\right|^2 \frac{(2\,{\rm Im}\tau)^2}{3} - k_W\left|H(\tau)\right|^2\right]$$

- $\Lambda_V^4 = \Lambda_W^6/\Lambda_K^2$ defines the energy scale of the potential
- $\hat{G}_2(\tau,\bar{\tau})$ is the non-holomorphic Eisenstein function of weight 2

Modular-invariant potentials

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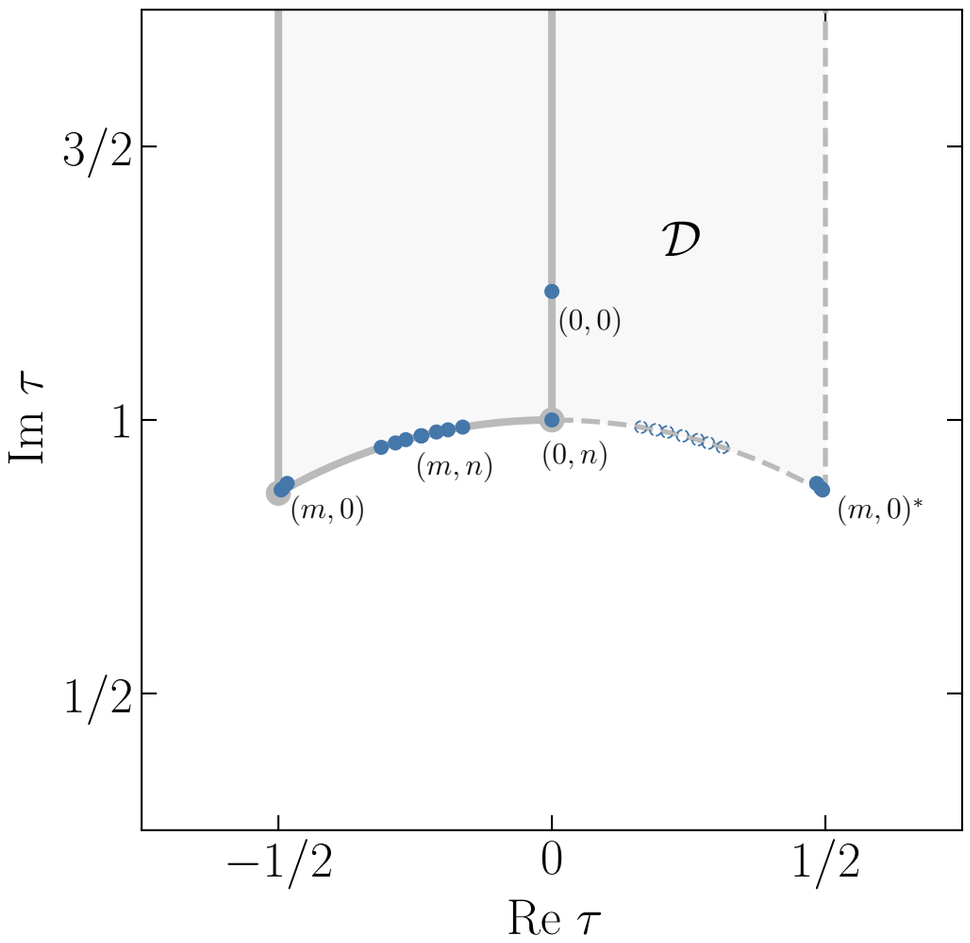
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In the global SUSY limit $k_W \to 0$ and

$$V(\tau,\bar{\tau}) = \frac{4}{3}\Lambda_V^4\,({\rm Im}\tau)^2\left|H'(\tau)\right|^2$$

Vacuum structure

Novichkov, Penedo, Petcov, JHEP 2022

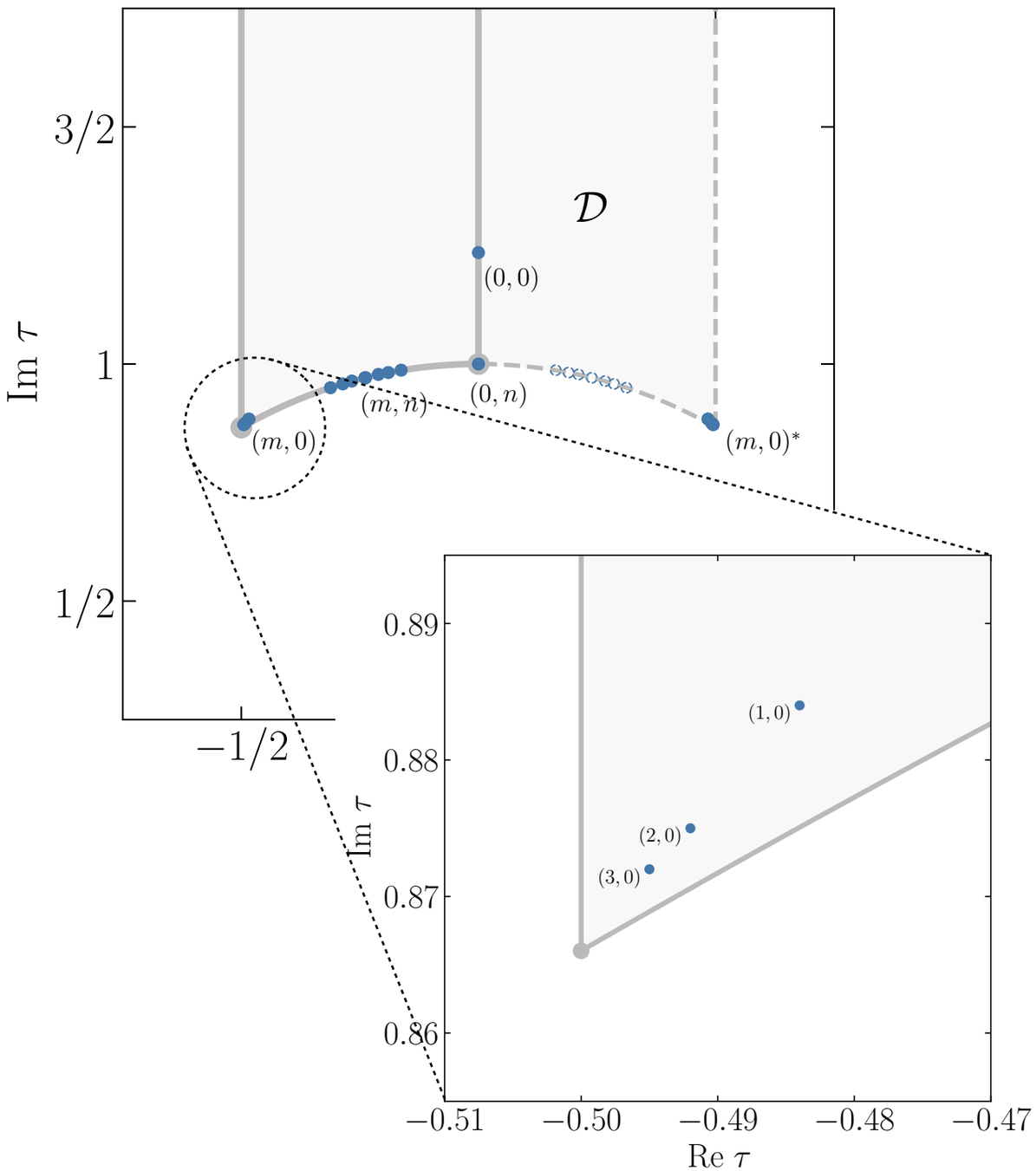


Cvetic et al., NPB 1991
and more recently
Novichkov, Penedo, Petcov, JHEP 2022
Leedom, Righi, Westphal, JHEP 2023
King, Wang, PRD 2024

For most values of (m, n), minima lie on the boundary of the fundamental domain D

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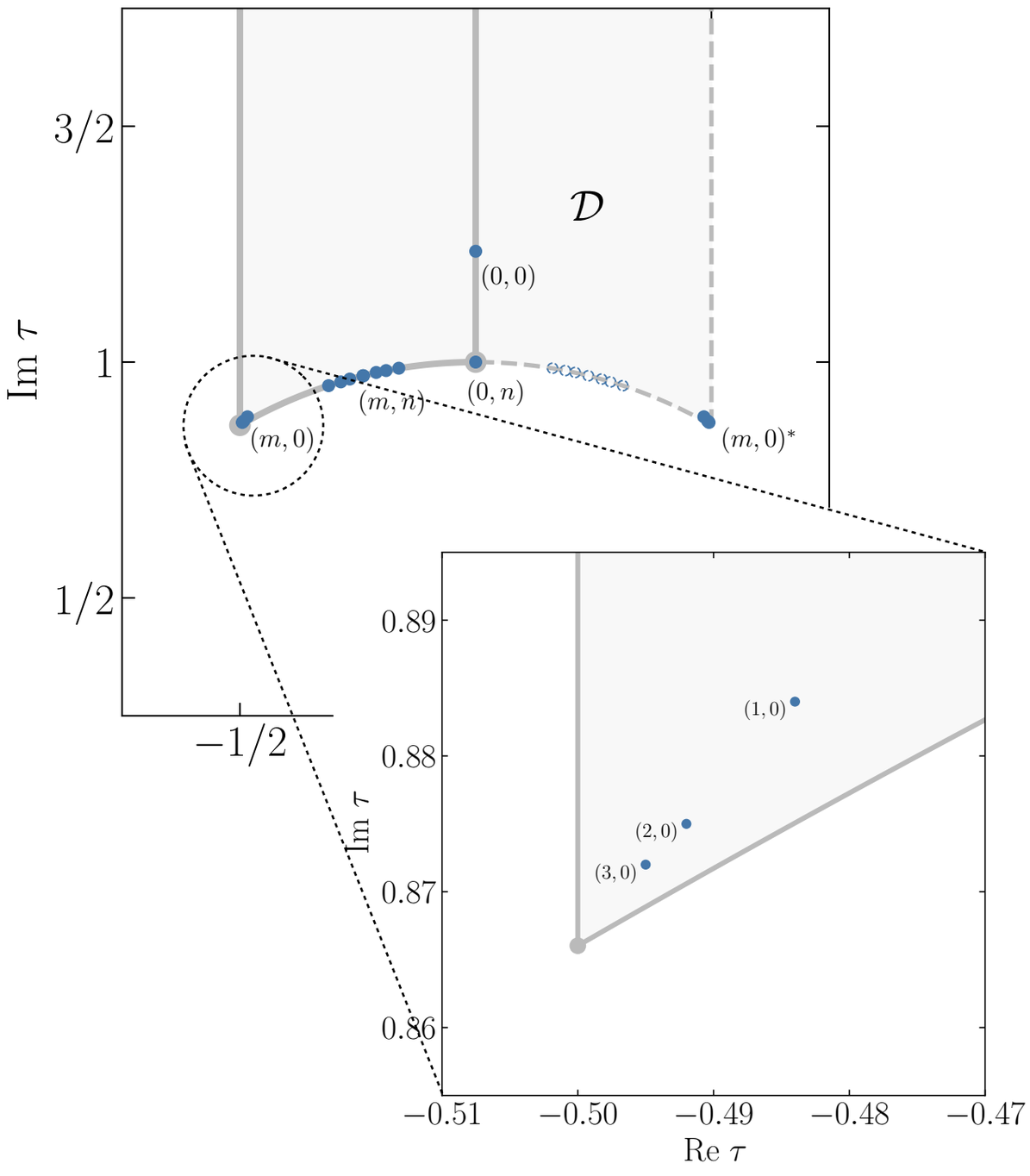
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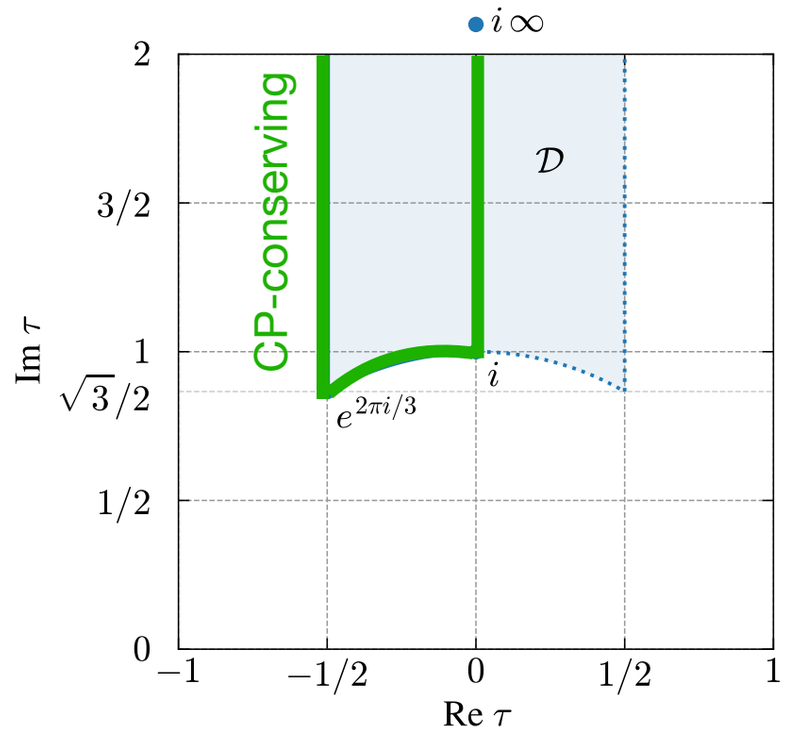


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The exception is $(m,0)$

Important for spontaneous CP violation



Novichkov, Penedo, Petcov, **AT**, JHEP 2019

Potential along the arc

Along the lower boundary of the fundamental domain

$$\tau = e^{i\theta} \quad \text{with} \quad \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}$$

Canonically normalised real scalar field φ

$$\mathscr{L}_{\text{kin}} = K_{\tau\bar{\tau}} \partial_\mu \tau \partial^\mu \bar{\tau} = \frac{3\Lambda_K^2}{4} \frac{\partial_\mu \theta \partial^\mu \theta}{\sin^2 \theta} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi$$

$$\varphi = \sqrt{\frac{3}{2}} \Lambda_K \ln \tan \frac{\theta}{2}$$

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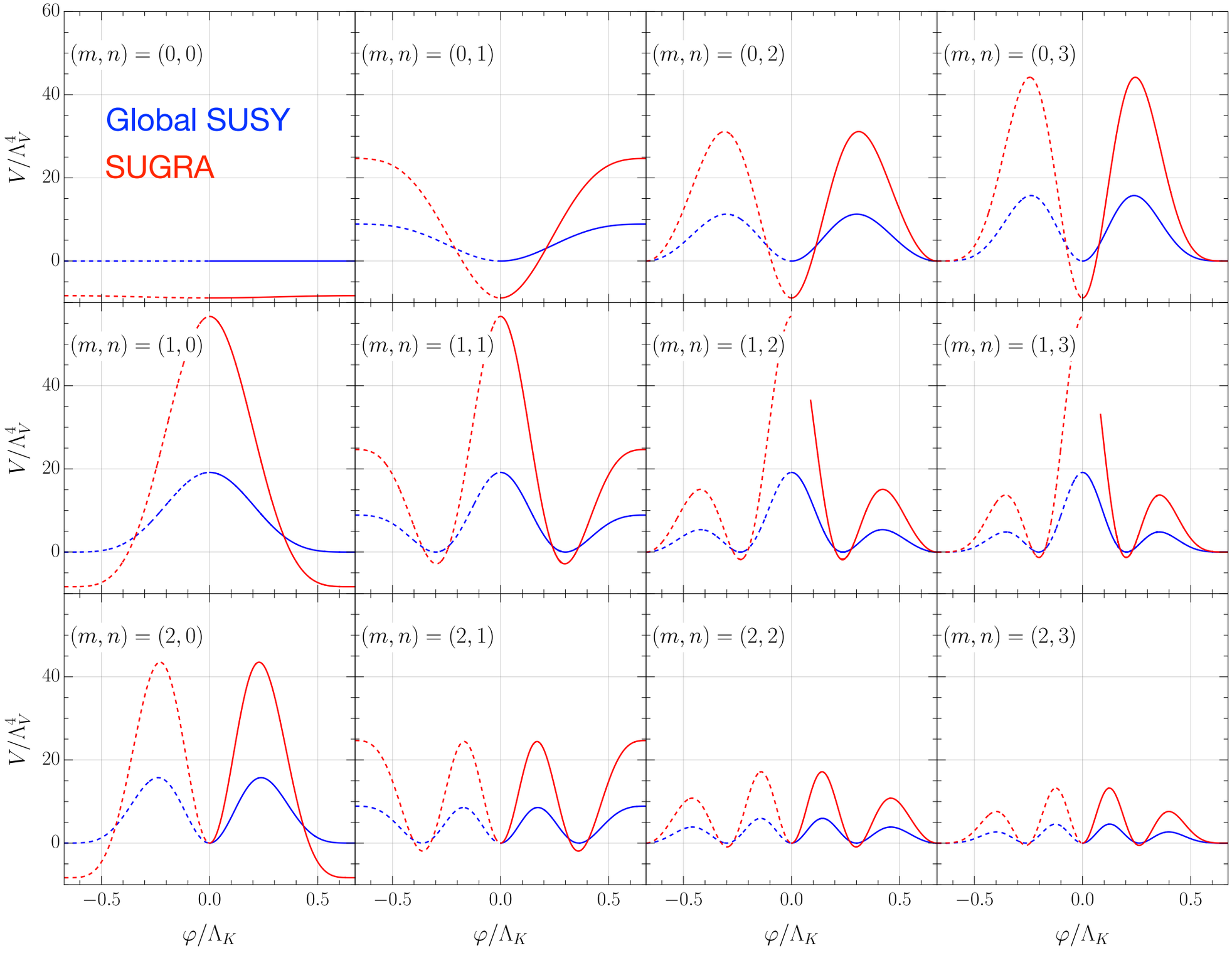
Under both S $(\tau \to -1/\tau)$ and CP $(\tau \to -\bar{\tau})$

$$\theta \to \pi - \theta \quad \Leftrightarrow \quad \varphi \to -\varphi \text{ (aka } Z_2)$$

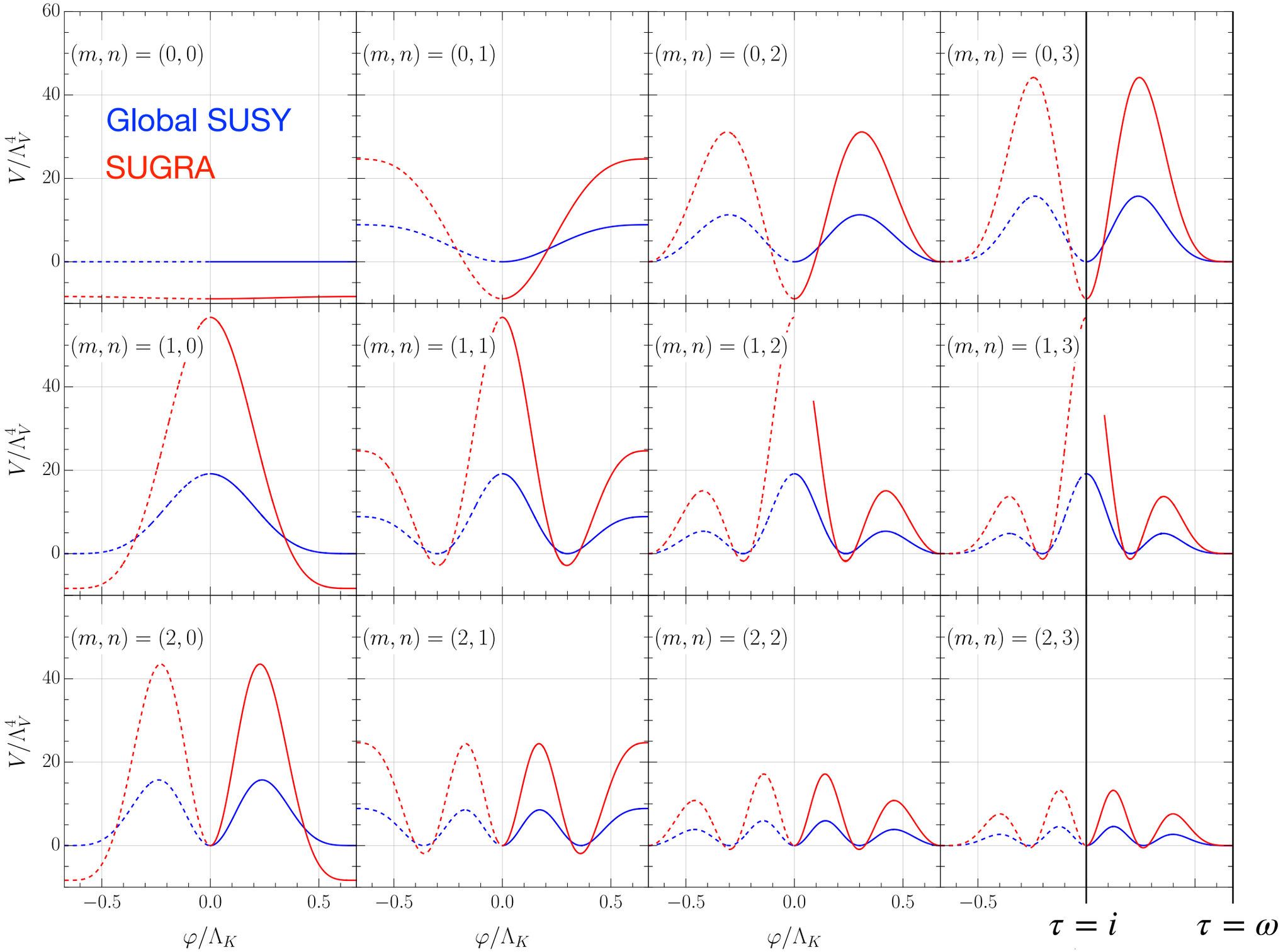
In a modular and CP-invariant theory, the potential along the arc

$$V(\varphi) = V(-\varphi)$$

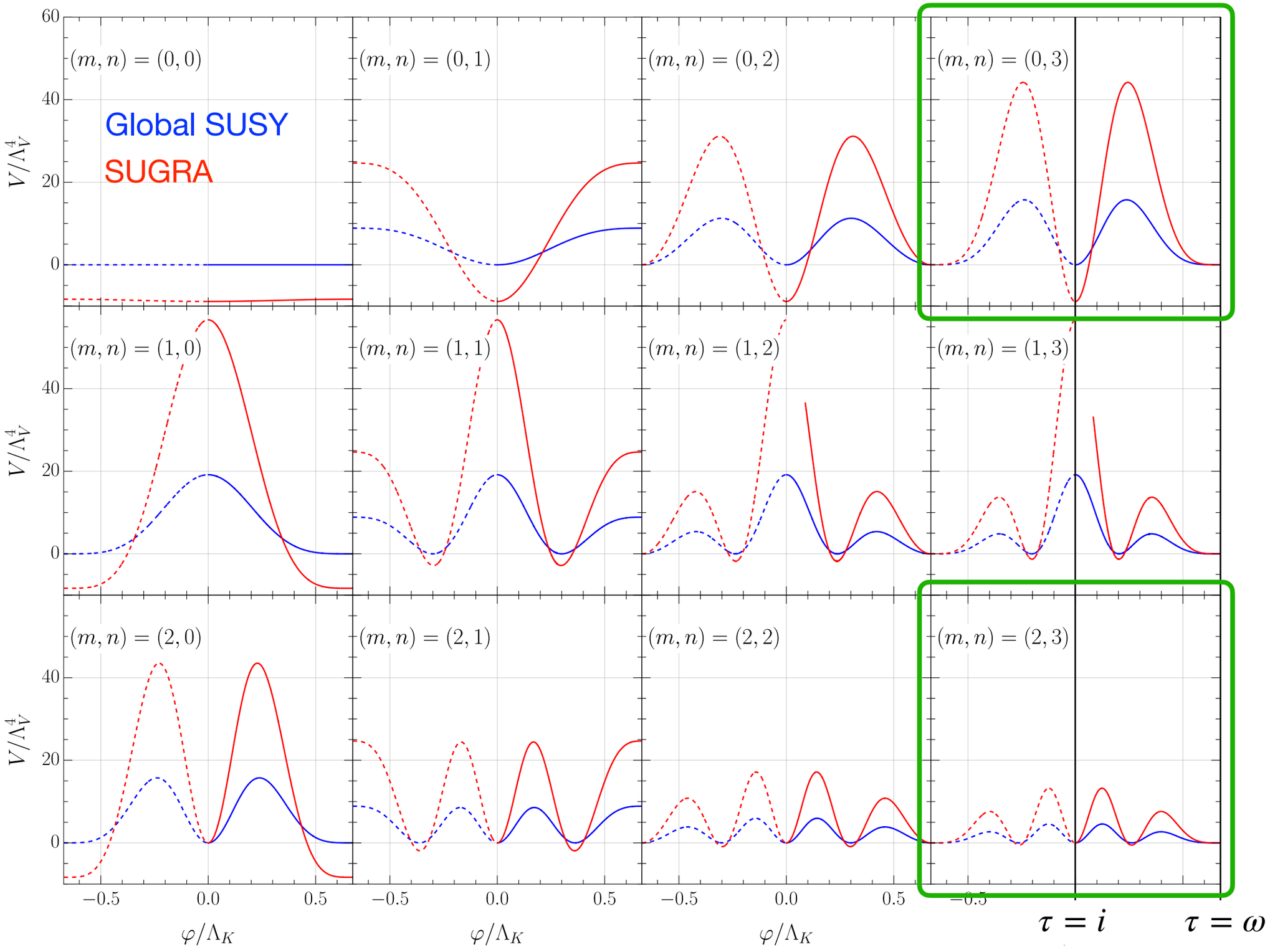
Potential along the arc



Potential along the arc



Potential along the arc



Dilaton contribution as a bias

Including the dilaton S (usually present in string-inspired theories)

$$K(\tau,\bar\tau,S,\bar S) = -3\Lambda_K^2 \ln\left(-i\tau+i\bar\tau\right) + \bar M_{\rm Pl}^2 K_{\rm dil}(S,\bar S)$$

$$W(\tau,S) = \Lambda_W^3 \frac{H(\tau)\,\Omega(S)}{\eta(\tau)^{2k_W}}$$

The scalar potential

$$V(\tau,\bar\tau) = \frac{\Lambda_V^4}{(2\,{\rm Im}\tau)^{k_W}\,|\,\eta(\tau)\,|^{4k_W}}\left[\left|iH'(\tau)+\frac{k_W}{2\pi}H(\tau)\hat G_2(\tau,\bar\tau)\right|^2\frac{(2\,{\rm Im}\tau)^2}{3}+\frac{k_W}{3}(A-3)\,\left|H(\tau)\right|^2\right]$$

$$A = A(S,\bar S) = K_{\rm dil}^{S\bar S}\frac{|\,\Omega_S + K_{{\rm dil},S}\Omega\,|^2}{|\,\Omega\,|^2}$$ (We will treat it as a free parameter)

$V(\tau,\bar\tau)\geq 0$ as long as $A\geq 3$

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Global SUSY

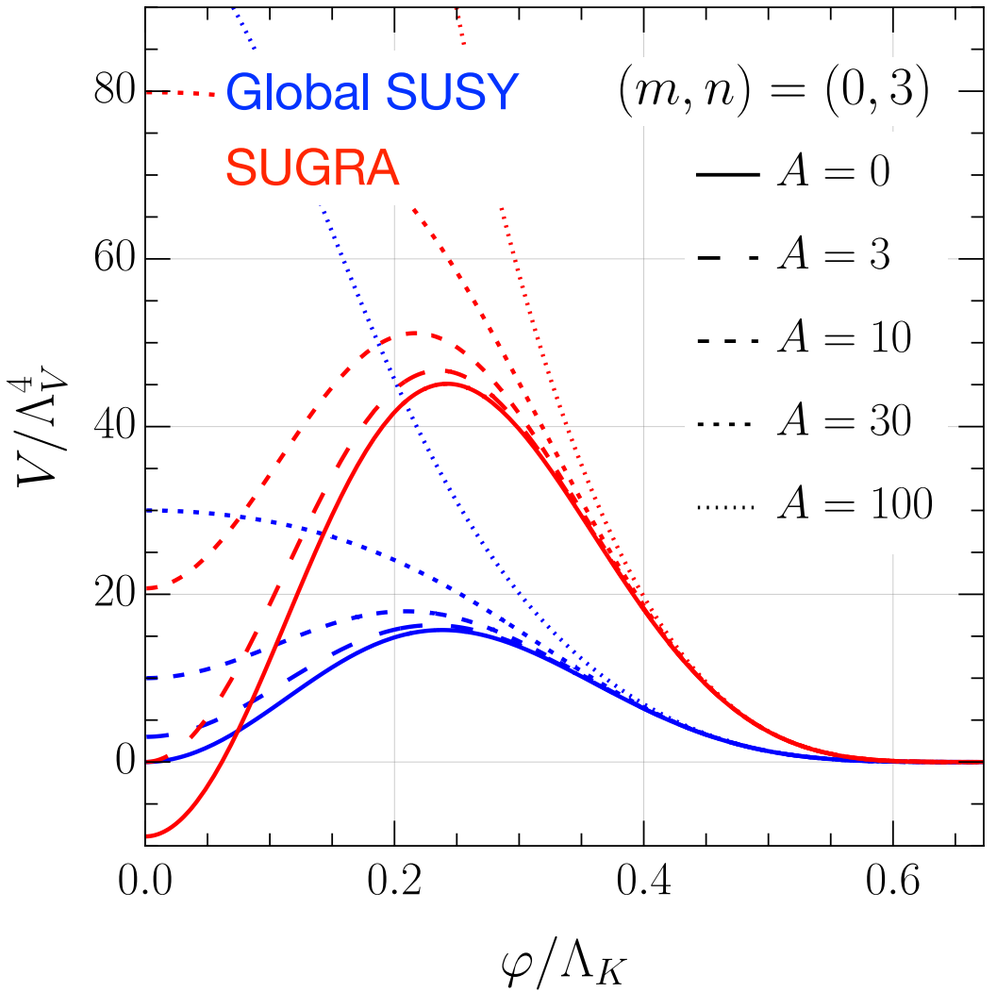
King, Wang, Zhou, JCAP 2025

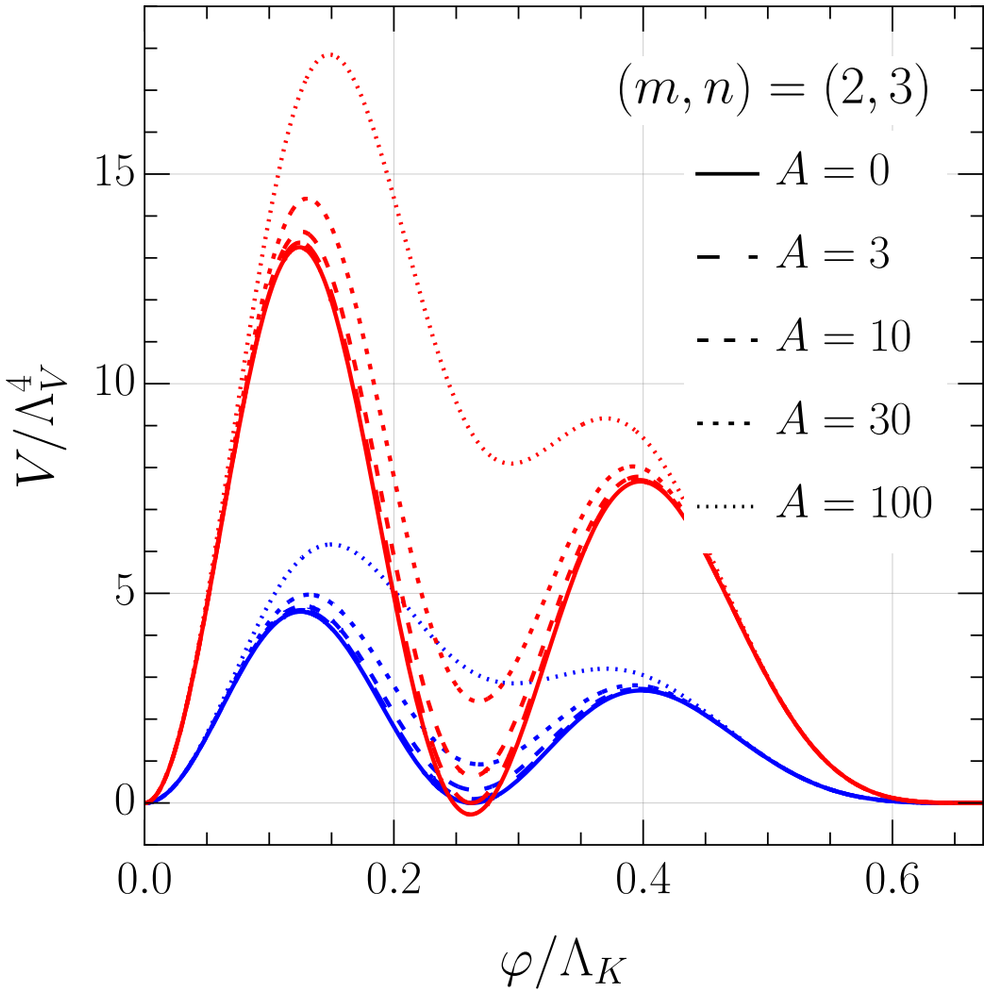
$k_W \to 0$ and A disappears, unless $A \sim 1/k_W \sim \bar M_{\rm Pl}^2/\Lambda_K^2$

$$V(\tau,\bar\tau) = \frac{4\Lambda_W^6}{3\Lambda_K^2}({\rm Im}\tau)^2\,|\,H'(\tau)\,|^2 + \underbrace{\frac{\Lambda_W^6}{\bar M_{\rm Pl}^2} A\,|\,H(\tau)\,|^2}$$

can serve as a bias
(modular-preserving, but SUSY breaking)

Dilaton contribution as a bias



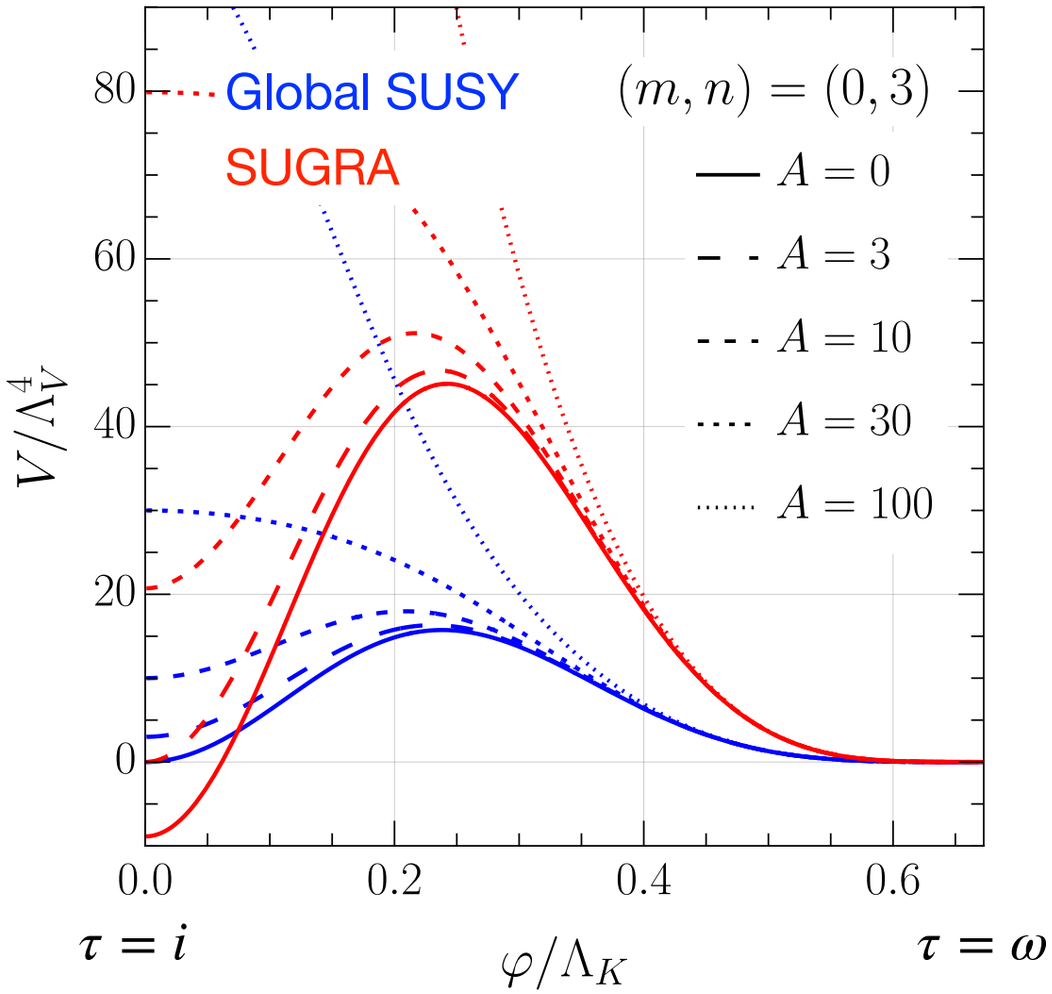


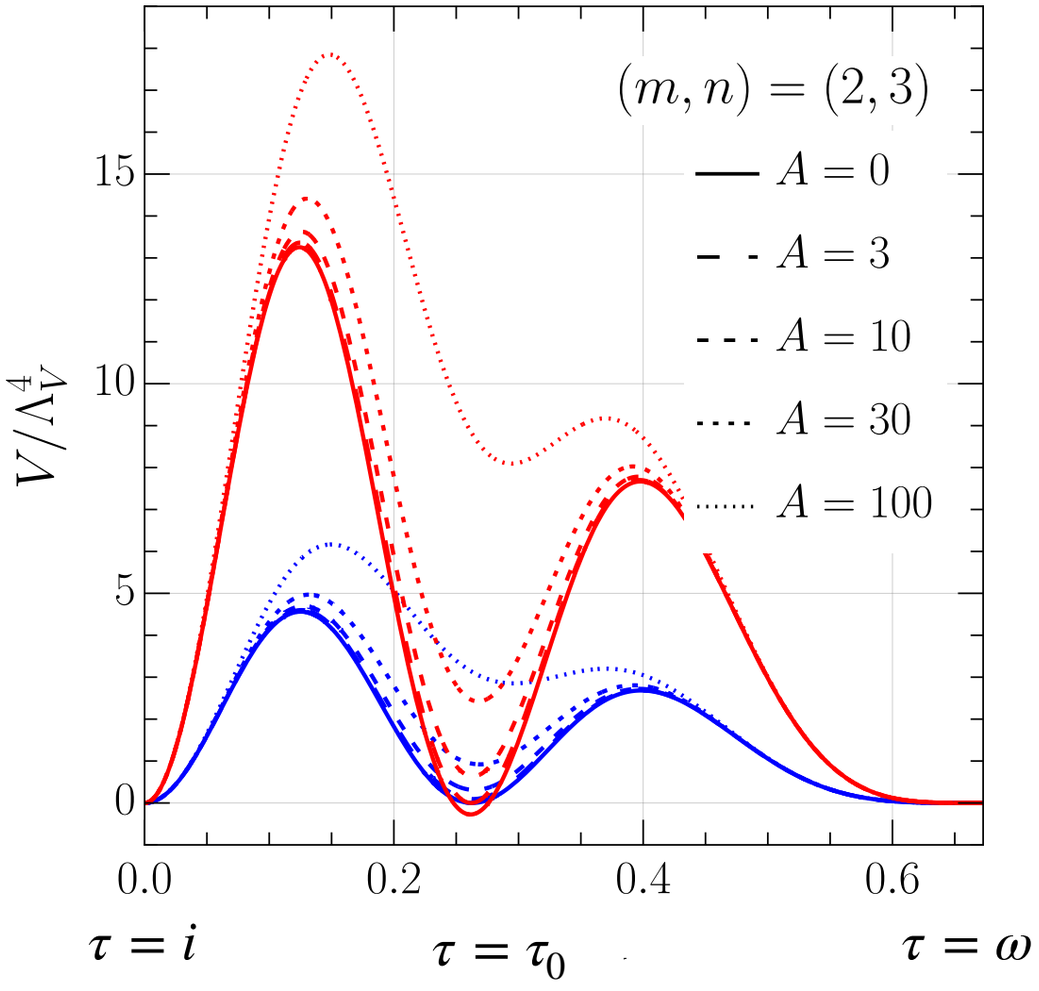
For global SUSY

$$V(\tau,\bar\tau) = \Lambda_V^4\left[\frac{4}{3}(\mathrm{Im}\tau)^2\,|\,H'(\tau)\,|^2 + \underbrace{\frac{\Lambda_K^2}{\bar M_{\rm Pl}^2}}_{k_W/3}\,A\,|\,H(\tau)\,|^2\right]$$

and the reported values are those of $A\,\Lambda_K^2/\bar M_{\rm Pl}^2 = A\,k_W/3$

Dilaton contribution as a bias



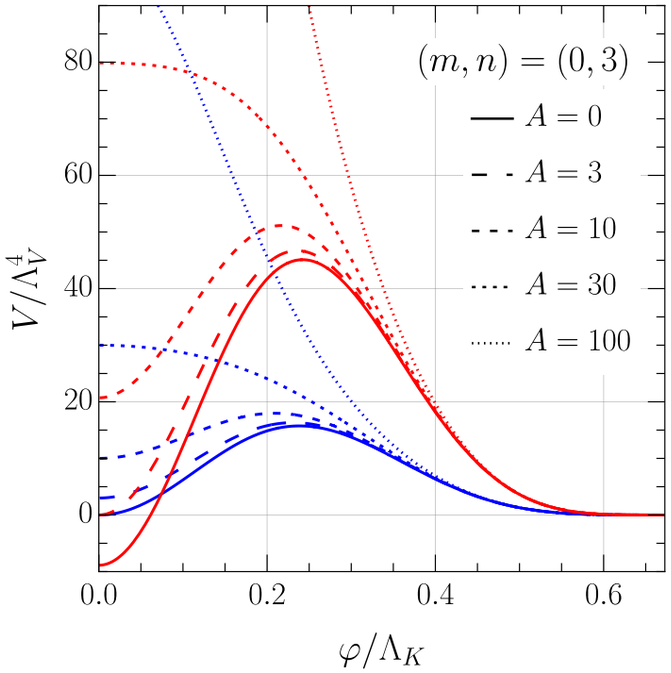


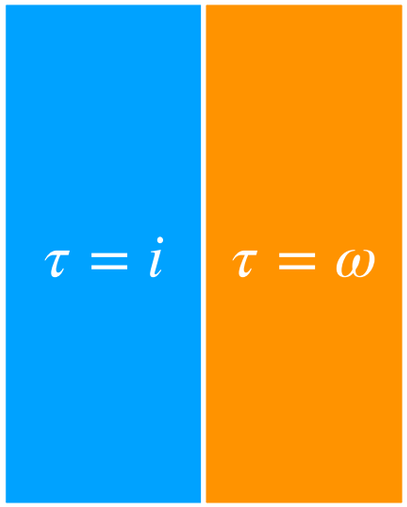
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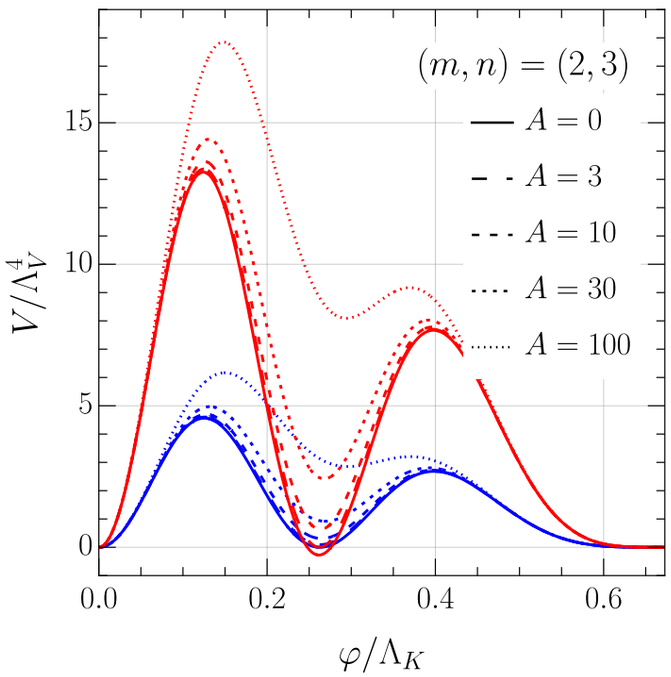
Modular domain walls

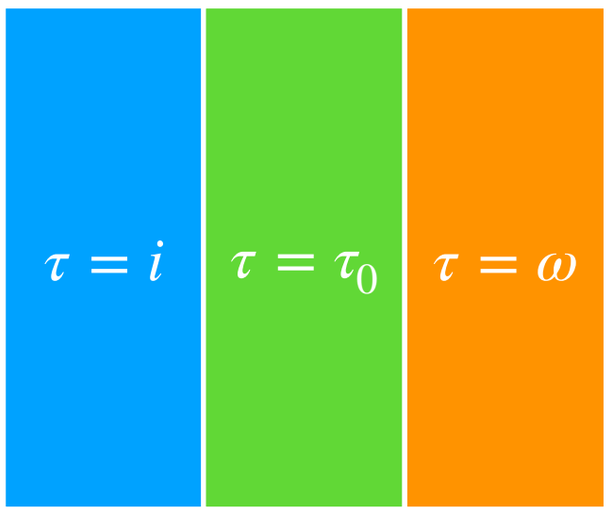




Cvetic, Quevedo, Rey, PRL 1991
Cvetic, Griffies, Rey, NPB 1992
Cvetic, Davis, PLB 1992

King, Wang, Zhou, JCAP 2025





Importantly, these vacua are *not* related by a symmetry transformation, unlike in the case of traditional discrete symmetries

Talk by Jessica Turner

Modular domain walls in global SUSY

Static wall perpendicular to z-axis

Cvetic, Quevedo, Rey, PRL 1991
Cvetic, Griffies, Rey, NPB 1992
Cvetic, Davis, PLB 1992

Tension (energy per unit area)

$$\sigma = \int_{-\infty}^{+\infty} dz \, \left(K_{\tau\bar{\tau}} \partial_z \tau \partial_z \bar{\tau} + K^{\tau\bar{\tau}} \partial_\tau W \partial_{\bar{\tau}} \overline{W} \right) = \int_{-\infty}^{+\infty} dz \, K_{\tau\bar{\tau}} \left| \partial_z \tau - e^{i\alpha} K^{\tau\bar{\tau}} \partial_{\bar{\tau}} \overline{W} \right|^2 + 2 \, \mathrm{Re} \left(e^{-i\alpha} \Delta W \right)$$

Kinetic energy density

Potential energy density

$$\Delta W \equiv W(\tau(z = +\infty)) - W(\tau(z = -\infty))$$

Choosing the phase α s.t. $e^{-i\alpha} \Delta W = |\Delta W|$, the Bogomolnyi bound follows

$$\boxed{\sigma \geq 2 |\Delta W|}$$

It is saturated when τ satisfies the BPS equation (first-order ODE)

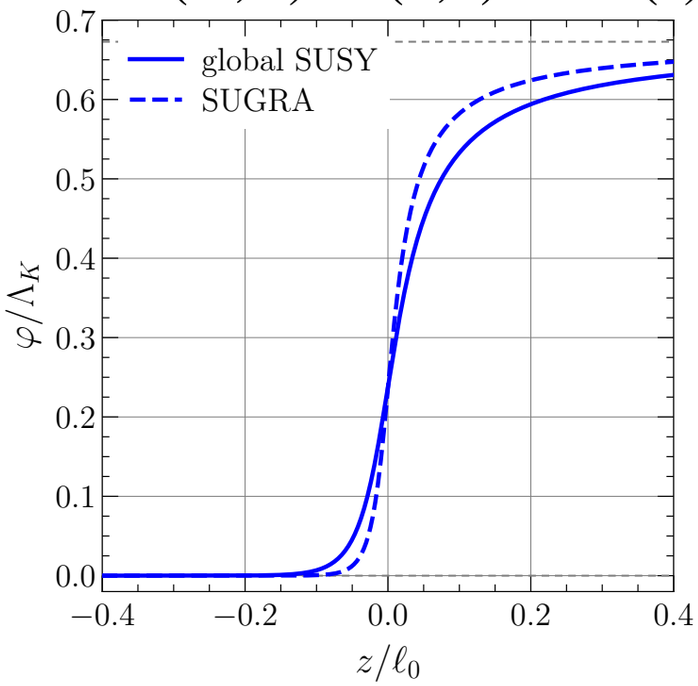
$$\partial_z \tau = e^{i\alpha} K^{\tau\bar{\tau}} \partial_{\bar{\tau}} \overline{W}$$

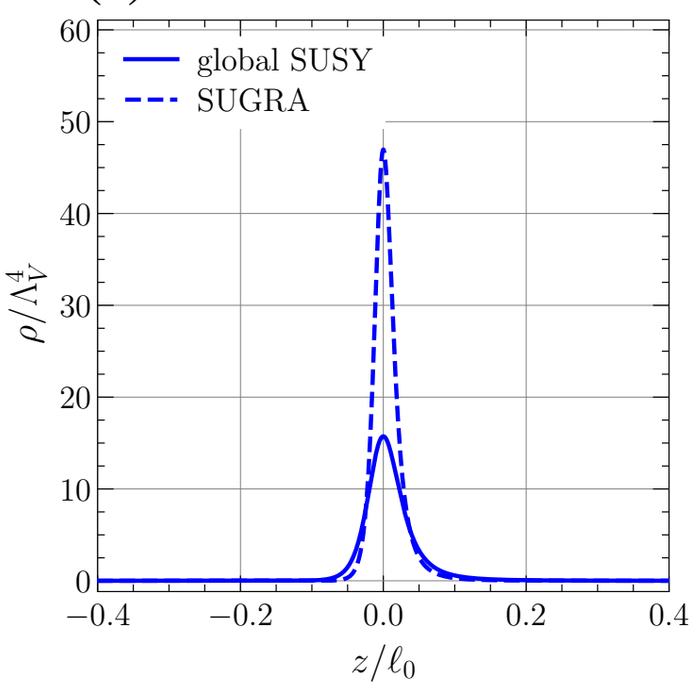
For $\varphi = \varphi(z)$, the EoM is $\partial_z^2 \varphi = \partial V / \partial \varphi$ and its first integral is $(\partial_z \varphi)^2 / 2 = V(\varphi)$ implying

$$\sigma = \int dz (\partial_z \varphi)^2 = \int d\varphi \sqrt{2V(\varphi)}$$

Modular domain walls

$(m,n) = (0,3)$ i.e. $H(\tau) = J(\tau)$: two vacua case





For global SUSY domain walls in the two vacua case see also King, Wang, Zhou, JCAP 2025

$$\ell_0 = \Lambda_K^2/\Lambda_W^3$$

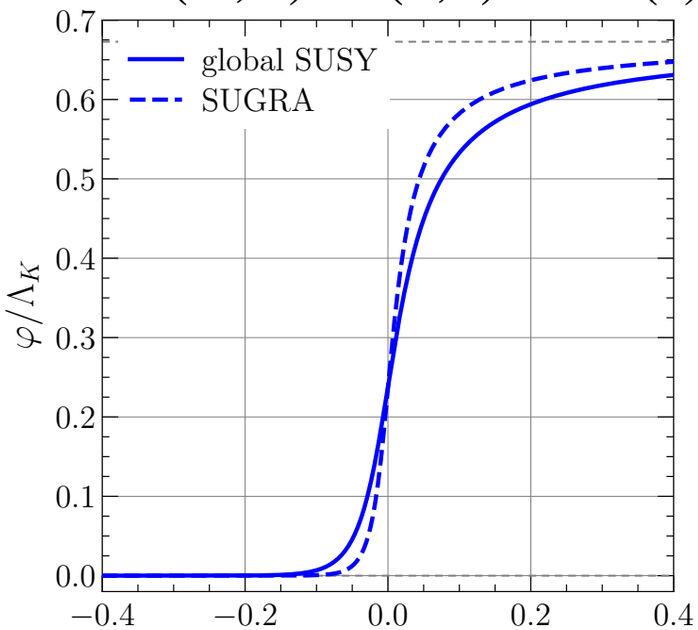
$$\rho = \begin{cases} \frac{1}{2}(\partial_z\varphi)^2 & \text{in SUSY} \\ \frac{1}{2}(\partial_r\varphi)^2 & \text{in SUGRA} \end{cases}$$

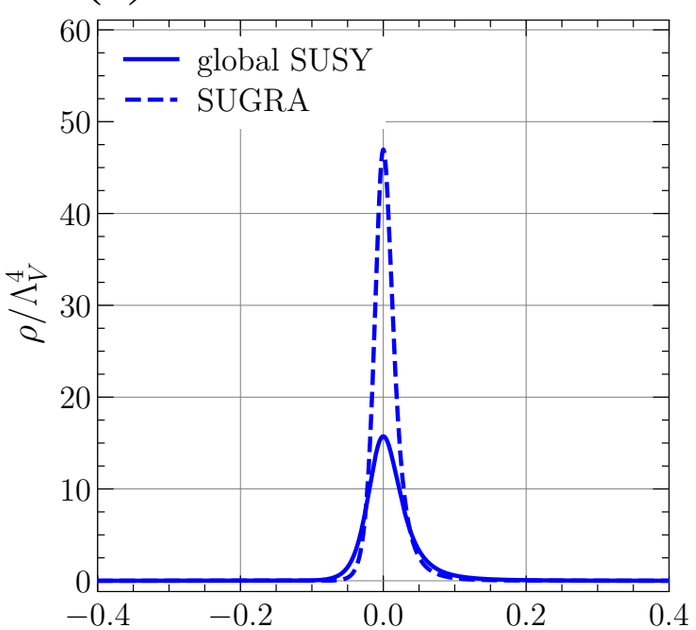
$dr = b(z)\,dz$ proper distance

In SUGRA, a Minkowski vacuum and an AdS vacuum may be connected by a static BPS wall since the pressure difference is balanced by the gravitational contribution encoded in the warp factor $b(z)$

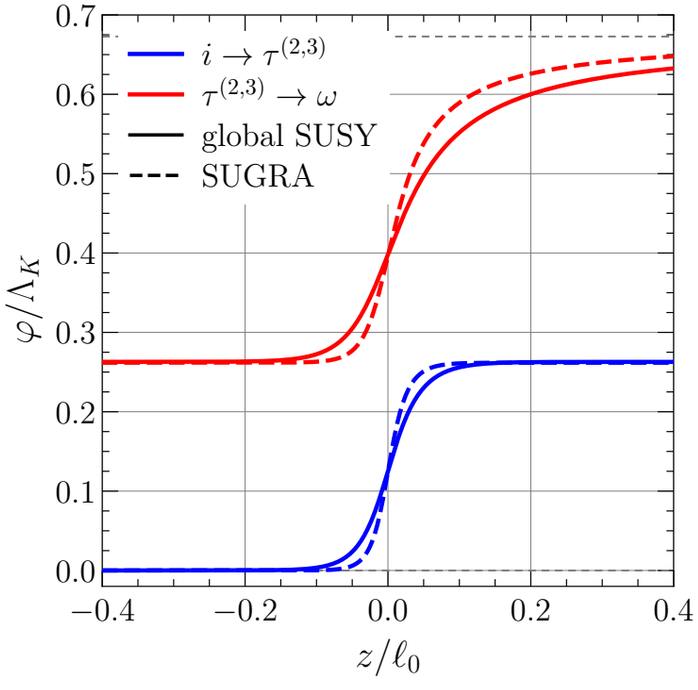
Modular domain walls

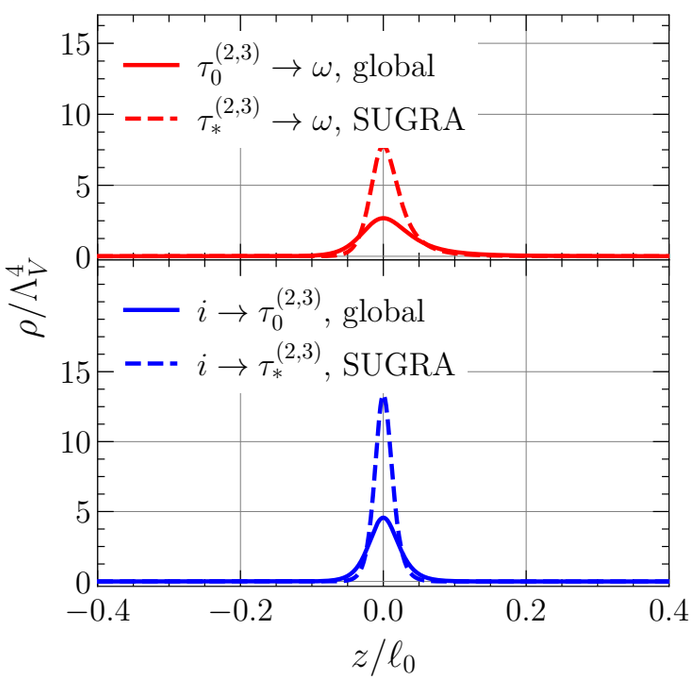
$(m,n) = (0,3)$ i.e. $H(\tau) = J(\tau)$: two vacua case





$(m,n) = (2,3)$ i.e. $H(\tau) = (J(\tau)-1)\,J(\tau)$: three vacua case





For global SUSY domain walls in the two vacua case see also King, Wang, Zhou, JCAP 2025

$$\ell_0 = \Lambda_K^2/\Lambda_W^3$$

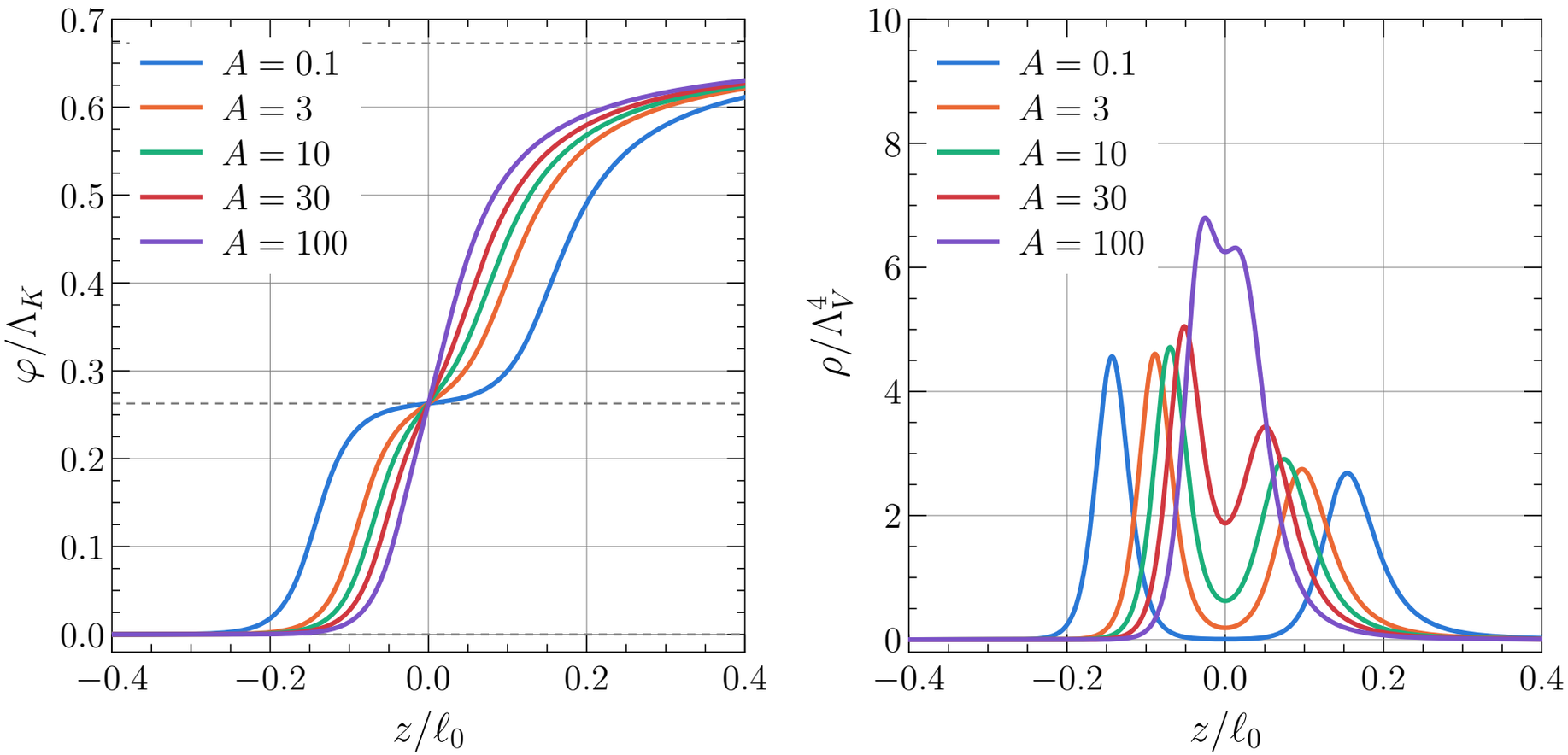
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$dr = b(z)\,dz$ proper distance

In SUGRA, a Minkowski vacuum and an AdS vacuum may be connected by a static BPS wall since the pressure difference is balanced by the gravitational contribution encoded in the warp factor $b(z)$

Biased modular domain walls

$(m,n) = (2,3)$ i.e. $H(\tau) = (J(\tau) - 1)\, J(\tau)$: three vacua case
with dilaton contribution $A \neq 0$ in global SUSY



Two walls $i\,|\,\tau_0$ and $\tau_0\,|\,\omega$ “recombine” into a single composite non-BPS wall $i\,|\,\omega$

The dilaton contribution $A \neq 0$ alone does not break the degeneracy $V(i) = V(\omega) = 0$,
and hence, does not drive annihilation of the composite $i\,|\,\omega$ walls. An additional bias is needed.

Gravitational waves from domain walls

$$V_{\rm bias,1} = \frac{A\Lambda_W^6}{\bar{M}_{\rm Pl}^2}\left|H(\tau_0)\right|^2$$ dilaton contribution

$$V_{\rm bias,2} = \left|V(i) - V(\omega)\right|$$ *ad hoc* bias (e.g. $\propto |J(\tau)|$, since $J(i) = 1$ and $J(\omega) = 0$)

In the unbiased BPS limit

$$\sigma_{i|\tau_0} = \sigma_{\tau_0|\omega} = 2\,|W(\tau_0)| \text{ since } W(i) = W(\omega) = 0$$

With a small $V_{\rm bias,1}$

$$\sigma_1 \approx \sigma_{i|\tau_0} = \sigma_{\tau_0|\omega}$$

With a small $V_{\rm bias,2}$

$$\sigma_2 \approx \sigma_{i|\tau_0} + \sigma_{\tau_0|\omega} \approx 2\sigma_1$$

During radiation domination, a scaling wall network annihilates when the bias pressure and the tension pressure balance, at

$$t_{\rm ann,1} \propto \frac{\sigma_1}{V_{\rm bias,1}} \quad \text{and} \quad t_{\rm ann,2} \propto \frac{\sigma_2}{V_{\rm bias,2}} \approx \frac{2\sigma_1}{V_{\rm bias,2}} \approx \frac{2V_{\rm bias,1}}{V_{\rm bias,2}}\, t_{\rm ann,1}$$

If $V_{\rm bias,2} \leq 2V_{\rm bias,1}$, then $t_{\rm ann,2} \geq t_{\rm ann,1}$

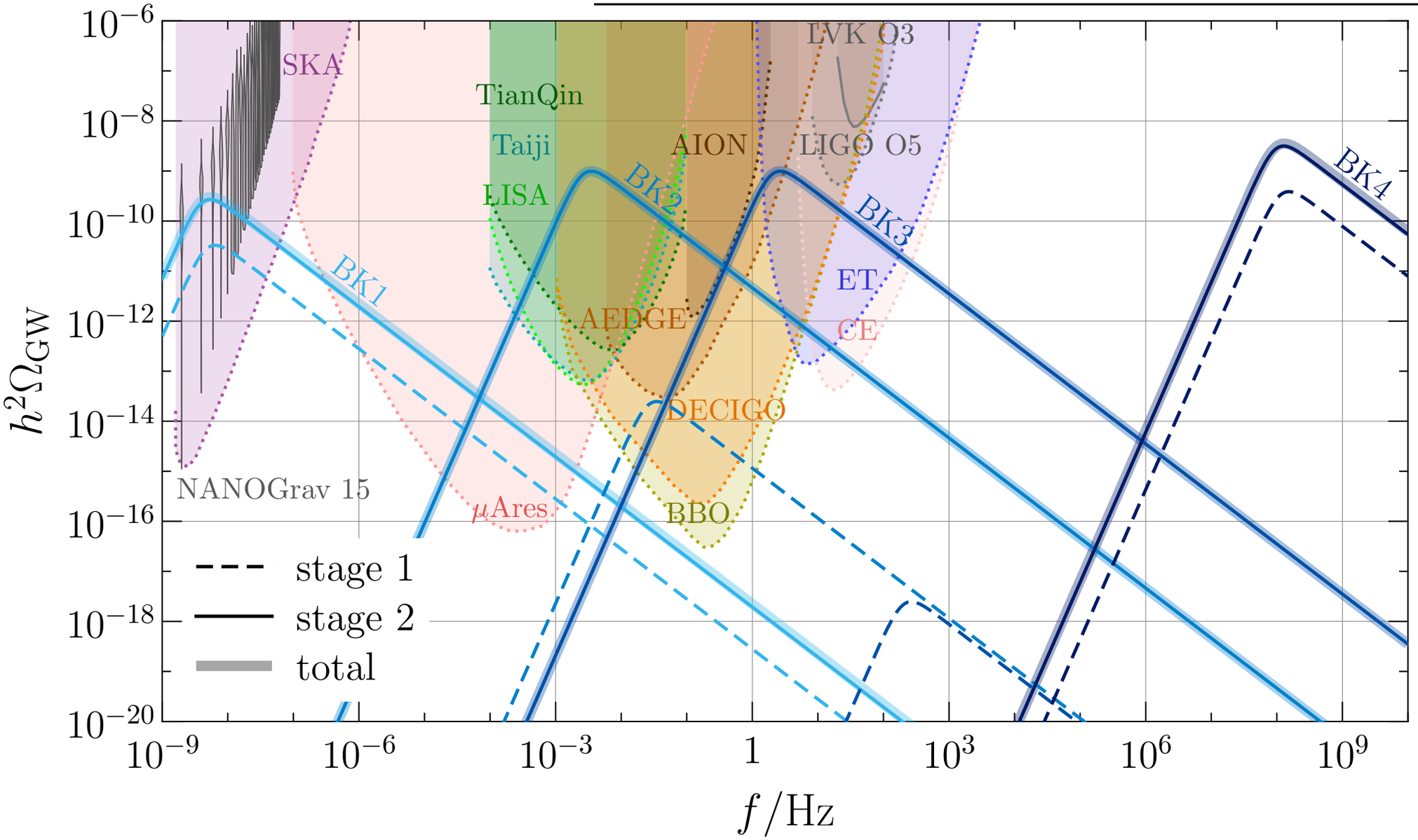
Gravitational waves from domain walls

Benchmarks

$$f_{p,1} \simeq 8.5 \times 10^{10}\ \text{Hz}\ \sqrt{A}\left(\Lambda_W/\bar{M}_{\rm Pl}\right)^{3/2}$$

$$\Omega_{p,1}h^2 \simeq 1.6 \times 10^{-5} A^{-2}$$

Benchmark	A	$\Lambda_W/\bar{M}_{\rm Pl}$	$V_{\rm bias,2}/V_{\rm bias,1}$
BK1	10^3	2.5×10^{-14}	1.4
BK2	2.5×10^4	3×10^{-10}	2×10^{-2}
BK3	2.5×10^6	2.5×10^{-8}	2×10^{-4}
BK4	2×10^2	4×10^{-3}	1.4



Conclusions

- Modular invariance is motivated by toroidal string theory compactifications
- It can be implemented consistently in an EFT
- It offers a promising approach to the flavour puzzle and the strong CP problem
- It also shapes the potential of the modulus τ
- Supergravity-motivated modular-invariant potentials have several minima and support formation of domain walls in the early Universe
- Biased modular domain walls can lead to testable gravitational waves providing a window to modular symmetry

Back-up slides

Domain walls between modular images

Since $H(\tau)$ has a non-trivial multiplier system, $W \to e^{i\alpha(\gamma)}W$, in particular

$$W(S\tau) = (-1)^m W(\tau) \quad \text{and} \quad W((ST)\tau) = e^{-2\pi i\, n/3} W(\tau)$$

The tension $\sigma = 2\,|\,\Delta W\,|$ can be non-zero for certain modular images

(m,n)	Walls between modular-inequivalent vacua	Walls between modular images
$(0,0)$	$V \equiv 0$	—
$(0,1)$	—	$i \vert (ST)i$
$(1,0)$	—	$\omega \vert S\omega$
$(1,1)$	—	$\tau_0 \vert S\tau_0,\ \tau_0 \vert (ST)\tau_0$
$(0, n \geq 2)$	$i \vert \omega$	—
$(m \geq 2, 0)$	$i \vert \omega$	—
$(1, n \geq 2)$	$\tau_0 \vert \omega$	$\tau_0 \vert S\tau_0$
$(m \geq 2, 1)$	$i \vert \tau_0$	$\tau_0 \vert (ST)\tau_0$
$(m \geq 2, n \geq 2)$	$i \vert \tau_0,\ \tau_0 \vert \omega$	—

Modular domain walls in SUGRA

The wall back-reacts on the metric

Cvetic, Griffies, Rey, NPB 1992
Cvetic, Davis, PLB 1992

$$ds^2 = b^2(z)\left(dt^2 - dx^2 - dy^2 - dz^2\right)$$

Warp factor $b(z)$

The BPS equation generalises as

$$\partial_z \tau = -\,\zeta b\, e^{K/(2\bar{M}_{\rm Pl}^2)}\,|\,W\,|\,K^{\tau\bar{\tau}}\overline{\left(\frac{D_\tau W}{W}\right)} \qquad (\zeta = \pm 1)$$

$$\partial_z \ln b = \zeta b\, e^{K/(2\bar{M}_{\rm Pl}^2)}\frac{|\,W\,|}{\bar{M}_{\rm Pl}^2}$$

The Bogomolnyi bound is

$$\sigma \geq 2\,|\,\Delta(\zeta Z)\,| \quad \text{with} \quad Z(\tau,\bar{\tau}) = e^{K/(2\bar{M}_{\rm Pl}^2)}\,|\,W\,|$$

Importantly, a Minkowski vacuum and an AdS vacuum may be connected by a static BPS wall since the pressure difference is balanced by the gravitational contribution encoded in $b(z)$

Gravitational waves from domain walls

Annihilation temperature, and peak frequency and amplitude today: Saikawa, Universe 2017

$$T_{\rm ann} \simeq 9.46\times 10^{17}\,{\rm GeV}\, C_{\rm ann}^{-1/2}\mathcal{S}^{-1/2}\left(\frac{g_*(T_{\rm ann})}{100}\right)^{-1/4}\left(\frac{\sigma}{\bar{M}_{\rm Pl}^3}\right)^{-1/2}\left(\frac{V_{\rm bias}}{\bar{M}_{\rm Pl}^4}\right)^{1/2}$$

$$f_p \simeq 1.53\times 10^{11}\,{\rm Hz}\, C_{\rm ann}^{-1/2}\mathcal{S}^{-1/2}\left(\frac{g_*(T_{\rm ann})}{100}\right)^{-1/12}\left(\frac{\sigma}{\bar{M}_{\rm Pl}^3}\right)^{-1/2}\left(\frac{V_{\rm bias}}{\bar{M}_{\rm Pl}^4}\right)^{1/2}$$

$$\Omega_p h^2 \simeq 8.63\times 10^{-7}\epsilon\mathcal{S}^4 C_{\rm ann}^2\left(\frac{g_*(T_{\rm ann})}{100}\right)^{-1/3}\left(\frac{\sigma}{\bar{M}_{\rm Pl}^3}\right)^{4}\left(\frac{V_{\rm bias}}{\bar{M}_{\rm Pl}^4}\right)^{-2}$$

First-stage scaling with A and Λ_W

$$T_{\rm ann,1} \simeq 5.29\times 10^{17}\ {\rm GeV}\ \sqrt{A}\left(\Lambda_W/\bar{M}_{\rm Pl}\right)^{3/2}$$

$$f_{p,1} \simeq 8.54\times 10^{10}\ {\rm Hz}\ \sqrt{A}\left(\Lambda_W/\bar{M}_{\rm Pl}\right)^{3/2}$$

$$\Omega_{p,1}h^2 \simeq 1.59\times 10^{-5}A^{-2}$$

GW spectrum (assuming broken power law)

$$h^2\Omega_{\rm GW}(f) \simeq \sum_{k=1}^{2} h^2\Omega_{p,k}\,\frac{4}{x_k^{-3}+3x_k}\,, \qquad x_k \equiv \frac{f}{f_{p,k}}$$

Modular group

Homogeneous modular group

$$\Gamma = \langle S,\, T \mid S^4 = (ST)^3 = I,\, S^2T = TS^2 \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry

Inhomogeneous modular group

$$\overline{\Gamma} = \langle S,\, T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{I, -I\}$$

In other words, $\mathrm{SL}(2,\mathbb{Z})$ matrices γ and $-\gamma$ are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \tau \xrightarrow{-\gamma} (-\gamma)\,\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

$$\gamma = -I \quad \Rightarrow \quad f(\tau) = (-1)^k f(\tau) \quad \Rightarrow \quad k \text{ is even}$$

Modular forms are periodic and admit q-expansions

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau+1) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i \tau}$$

Modular forms of weight k form a linear space $\mathscr{M}_k$ of finite dimension

$$\dim \mathscr{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k = 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \neq 2 \pmod{12} \end{cases}$$

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$$f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma$$

k is weight, a non-negative even integer

Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

Each modular form of weight k can be written as a polynomial in E_4 and E_6

$$f(\tau) = \sum_{a,b\geq 0} c_{ab}\, E_4^a(\tau)\, E_6^b(\tau) \qquad \text{with} \qquad 4a + 6b = k$$

Modular forms

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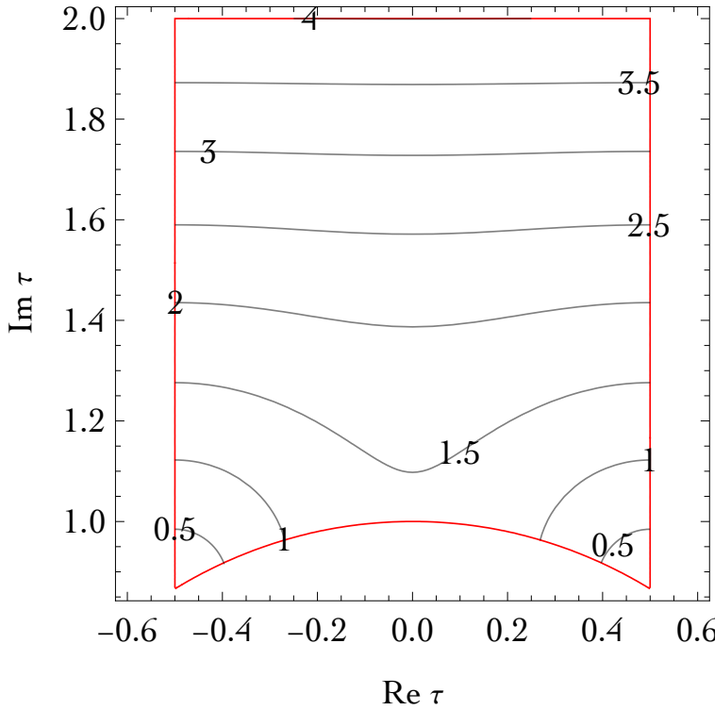
Modular weight k	0	2	4	6	8	10	12	14
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

E4 and E6

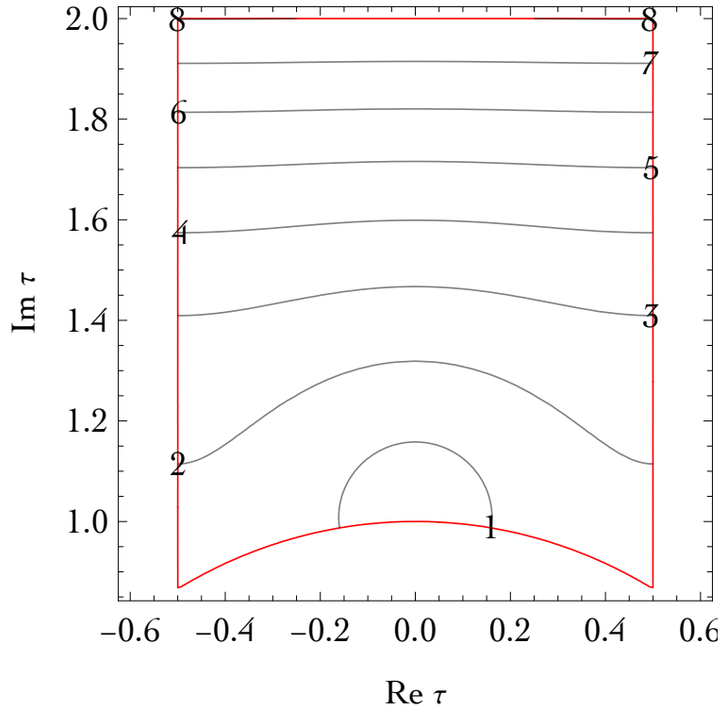
$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \mathcal{O}(q^5)$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 + \mathcal{O}(q^5)$$

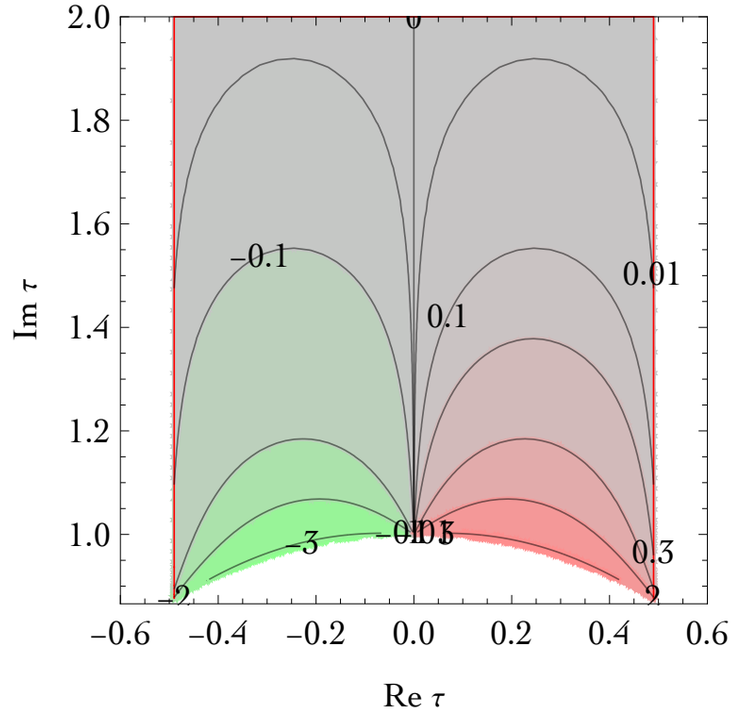
$|(\mathrm{Im}\ \tau)^2 E_4(\tau)|$



$|(\mathrm{Im}\ \tau)^3 E_6(\tau)|$



$\arg E_4^3/E_6^2$



Modular-invariant SUSY theories

Under modular transformations $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Ferrara et al., PLB **225** (1989) 363
Ferrara et al., PLB **233** (1989) 147
Feruglio, 1706.08749

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} & \text{modulus is promoted to a (dimensionless) superfield} \\ S \to S & \text{dilaton} \\ \Phi \to \left(c\tau + d\right)^{-k_\Phi} \Phi & \text{matter supermultiplets} \\ V \to V & \text{vector supermultiplets} \end{cases}$$

k_Φ is modular weight of Φ

Modular symmetry acts non-linearly on field space

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modulus is promoted to a (dimensionless) superfield

dilaton

matter supermultiplets

vector supermultiplets

k_Φ is modular weight of Φ

Modular symmetry acts non-linearly on field space

Modular invariance of the action requires

$$\begin{cases} K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) \to K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) + f_K(\tau, \Phi) + \overline{f_K}\left(\tau^\dagger, \Phi^\dagger\right) \\ W(\tau, \Phi) \to W(\tau, \Phi) \\ f_a(S, \tau) \to f_a(S, \tau) + \dfrac{1}{8\pi^2} \sum\limits_\Phi 2T_a(\Phi)\, k_\Phi \ln(c\tau + d) \end{cases}$$

anomaly term

$T_a(\Phi)$ is Dynkin index of Φ gauge rep

Konishi, Shizuya, Nuovo Cim. A **90** (1985) 111
Arkani-Hamed, Murayama, hep-th/9707133

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_\Phi \frac{\Phi^\dagger e^{2V}\Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_\Phi}}$$

Modular-invariant SUSY theories

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$$K = -h^2 \ln\left(-i\tau + i\tau^\dagger\right) + \sum_\Phi \frac{\Phi^\dagger e^{2V}\Phi}{\left(-i\tau + i\tau^\dagger\right)^{k_\Phi}}$$

Superpotential

$$W = U_i^c\, Y^u_{ij}(\tau)\, Q_j\, H_u + D_i^c\, Y^d_{ij}(\tau)\, Q_j\, H_d + E_i^c\, Y^e_{ij}(\tau)\, L_j\, H_d + \frac{1}{2\Lambda_L} L_i\, C^\nu_{ij}(\tau)\, L_j\, H_u\, H_u$$

τ-dependent Yukawa couplings (and Majorana mass)

$$Y^u_{ij}(\tau) \to \left(c\tau + d\right)^{k_{Y^u_{ij}}} Y^u_{ij}(\tau) \qquad \text{with} \qquad k_{Y^u_{ij}} = k_{U^c_i} + k_{Q_j} + k_{H_u}$$

(similarly for Y^d, Y^e, C^ν)

are modular forms!

Modular-invariant SUSY theories

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$$W = U_i^c\, Y_{ij}^u(\tau)\, Q_j\, H_u + D_i^c\, Y_{ij}^d(\tau)\, Q_j\, H_d + E_i^c\, Y_{ij}^e(\tau)\, L_j\, H_d + \frac{1}{2\Lambda_L} L_i\, C_{ij}^\nu(\tau)\, L_j\, H_u\, H_u$$

τ-dependent Yukawa couplings (and Majorana mass)

$$Y_{ij}^u(\tau) \to \left(c\tau + d\right)^{k_{Y_{ij}^u}} Y_{ij}^u(\tau) \qquad \text{with} \qquad k_{Y_{ij}^u} = k_{U_i^c} + k_{Q_j} + k_{H_u}$$

(similarly for Y^d, Y^e, C^ν)

are modular forms!

$$\det\left(Y^u Y^d\right) \to \left(c\tau + d\right)^{k_{\det}} \det\left(Y^u Y^d\right)$$

$$k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

Modular invariance and CP

Fields

$$\tau \xrightarrow{\text{CP}} -\tau^\dagger \qquad \text{and} \qquad \Phi \xrightarrow{\text{CP}} \Phi^\dagger$$

Modular forms

$$Y(\tau) \xrightarrow{\text{CP}} Y(-\tau^*) = Y(\tau)^*$$

Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

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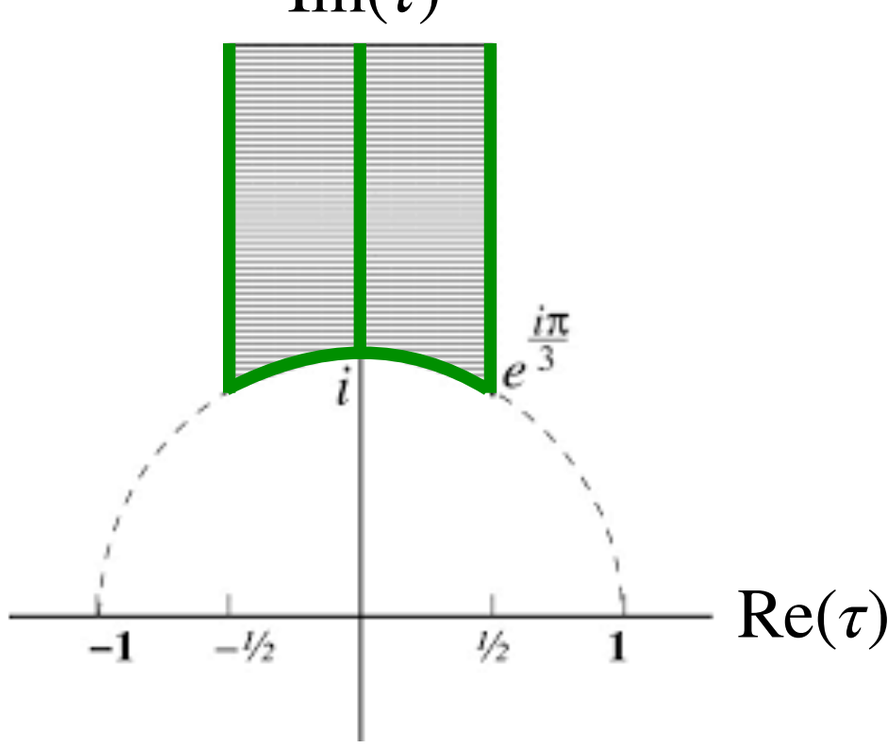
CP-conserving values of τ

$\tau \xrightarrow{\text{CP}} -\tau^* = \gamma\tau$ (goes to itself up to γ)

1. $\tau = iy \xrightarrow{\text{CP}} iy$

2. $\tau = -\frac{1}{2} + iy \xrightarrow{\text{CP}} \frac{1}{2} + iy = T\tau$

3. $\tau = e^{i\varphi} \xrightarrow{\text{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

Finite modular groups

Infinite normal subgroups of $\mathrm{SL}(2,\mathbb{Z})$, $N = 2\,,\ 3\,,\ 4\,,\ldots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

Principal congruence subgroups of the modular group

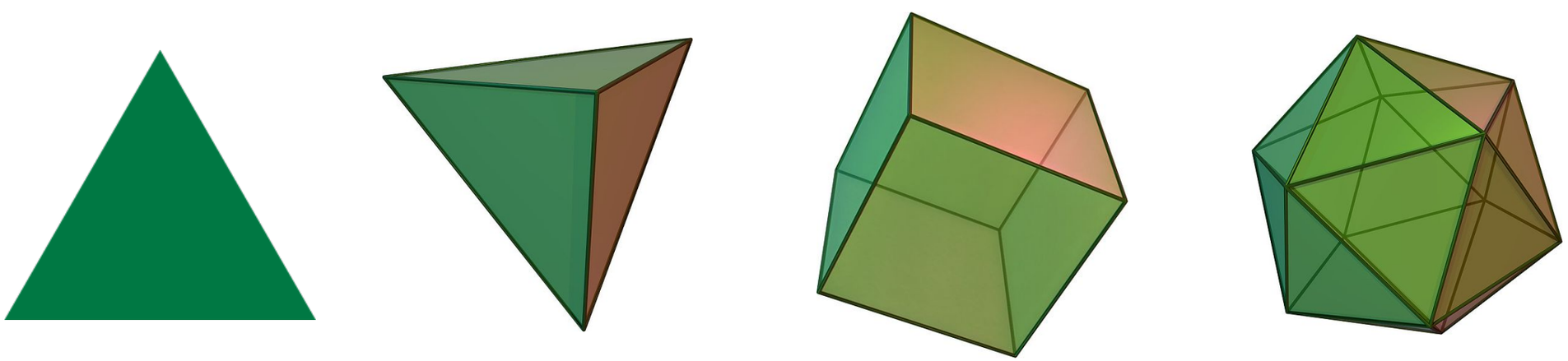
$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N)\,, \quad N > 2$$

Finite modular groups

$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$$

$$\Gamma_N = \left\langle S,\, T \mid S^2 = (ST)^3 = T^N = I \right\rangle\,, \quad N = 2\,,\ 3\,,\ 4\,,\ 5$$

$\Gamma_2 \cong S_3$ $\Gamma_3 \cong A_4$ $\Gamma_4 \cong S_4$ $\Gamma_5 \cong A_5$



Theories based on finite modular groups

$\mathcal{N} = 1$ rigid SUSY matter action

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\; K(\tau, \overline{\tau}, \psi, \overline{\psi}) + \int \mathrm{d}^4x\, \mathrm{d}^2\theta\; W(\tau, \psi) + \int \mathrm{d}^4x\, \mathrm{d}^2\overline{\theta}\; \overline{W}(\overline{\tau}, \overline{\psi})$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \psi_i \to \left(c\tau + d\right)^{-k_i} \rho_i(\tilde{\gamma})\, \psi_i \end{cases} \Rightarrow \begin{cases} W(\tau, \psi) \to W(\tau, \psi) \\ K(\tau, \overline{\tau}, \psi, \overline{\psi}) \to K(\tau, \overline{\tau}, \psi, \overline{\psi}) + f_K(\tau, \psi) + \overline{f_K}(\overline{\tau}, \overline{\psi}) \end{cases}$$

unitary representation of Γ_N

Feruglio, 1706.08749

$$W(\tau, \psi) = \sum_n \sum_{\{i_1, \ldots, i_n\}} \sum_s g_{i_1 \ldots i_n, s} \left(Y_{i_1 \ldots i_n, s}(\tau)\, \psi_{i_1} \ldots \psi_{i_n} \right)_{\mathbf{1}, s}$$

$$Y(\tau) \xrightarrow{\gamma} (c\tau + d)^{k_Y} \rho_Y(\tilde{\gamma})\, Y(\tau)$$

$$k_Y = k_{i_1} + \ldots + k_{i_n}$$

$$\rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_n} \supset \mathbf{1}$$

Yukawa couplings are modular forms!