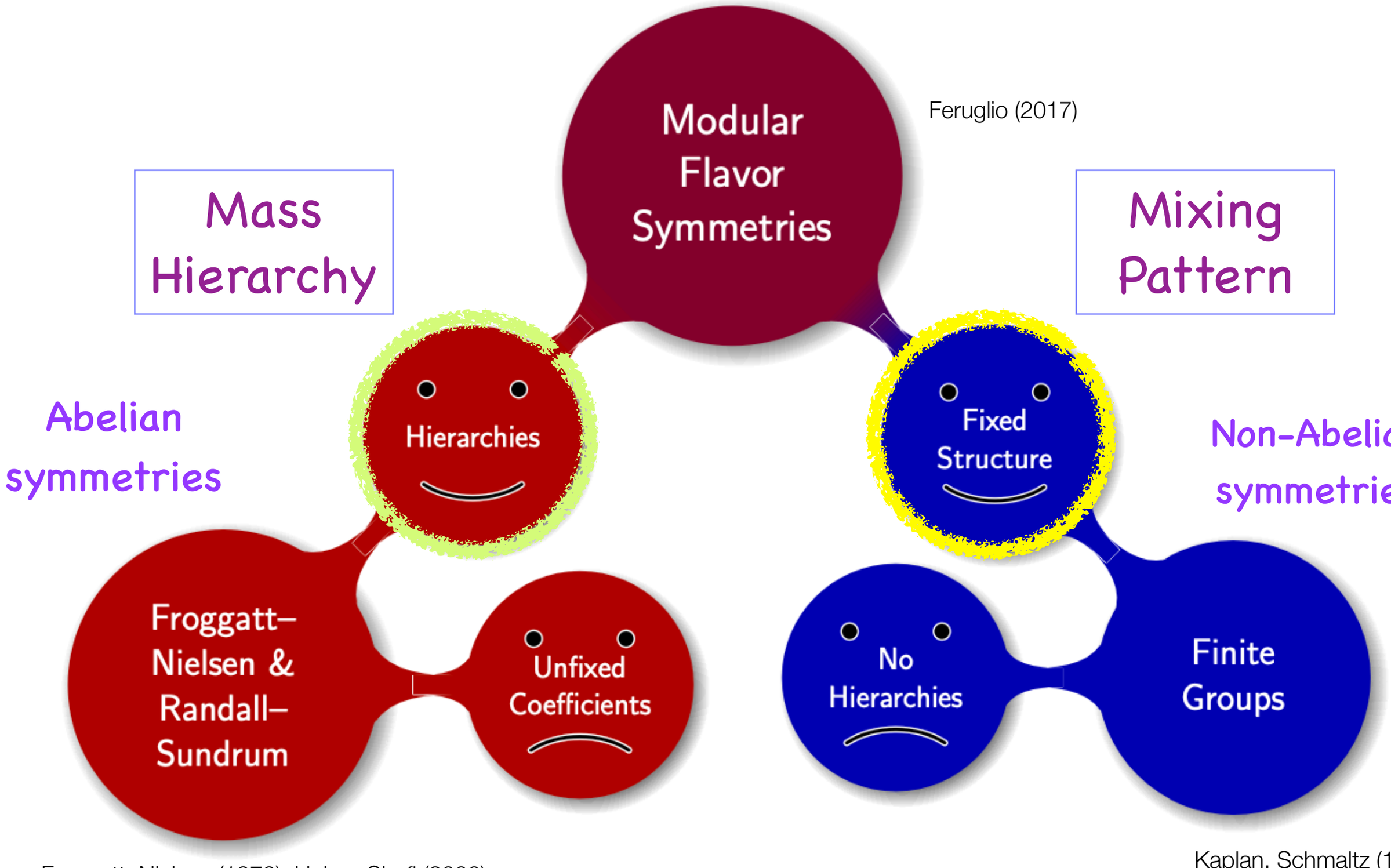
Modular Symmetries, Mass Hierarchy, and GW Probe of Flavor

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DISCRETE 2026, Bratislava, Slovakia, Nov 15, 2026

Theories of Flavor



Feruglio (2017)

Froggatt, Nielsen (1979); Huber, Shafi (2000)

Kaplan, Schmaltz (1993)

Modular Flavor Symmetries

Feruglio (2017)

- Imposing modular symmetry Γ on the Lagrangian:

$$\mathscr{L} \supset \sum Y_{i_1,\, i_2,\, \ldots..\, i_n} \Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_n}$$

see talks by Ferruccio Feruglio, Arsenii Titov, Salvador Centelles Chuliá

$$\tau \xrightarrow{\gamma} \gamma\tau := \frac{a\tau + b}{c\tau + d}\,,$$

$$\Phi_j \xrightarrow{\gamma} (c\tau + d)^{k_j} \rho_{\mathbf{r}_j}(\gamma)\, \Phi_j\,, \quad \text{where } \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

k_i : integers

representation matrix of Γ_N

- Yukawa Couplings = Modular Forms at level "N" w/ weight "k"

$$f_i(\gamma\tau) = (c\tau + d)^{-k} [\rho_N(\gamma)]_{ij} f_j(\tau)$$

$k = k_{i_1} + k_{i_2} + \ldots + k_{i_n}$

representation matrix of Γ_N

Predictive Power of Modular Symmetries

- Modular Invariance:

  modular weights + representation assignments ⇒ highly constrained framework

  - Much fewer input parameters ⇒ predictions consistent with data

- This Talk: Two generic predictions from properties of modular forms

  - Invariants ⇒ robust sum rules among mixing parameters:

    modular invariance (even τ-independent), RG invariant

    MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

    MCC, S. Perez, M. Ratz (2026)

  - Near-critical behavior of mass matrices ⇒ Mass hierarchy at near critical points, determined by modular weights MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

Predictive Power of Modular Symmetries

- Ingredients
  - Modular invariance
  - Holomorphicity
  - Finiteness

- However, typical observables are not holomorphic, e.g.

$$\mathscr{W} = \frac{\mathscr{M}(\tau)}{2}\Phi^2$$

$$\mathscr{K} = \frac{1}{(-i\tau + i\overline{\tau})^{k_\Phi}}\overline{\Phi}\Phi$$

$$m_{physical} = m_{physical}(\overline{\tau}, \tau)$$

$$= |\mathscr{M}(\tau)|(-i\tau + i\overline{\tau})^{k_\Phi}$$

Are there observables fulfilling the three properties?

Holomorphic Observables

- Typical model

$$\mathscr{W}_{lepton} = Y_e^{ij} L_i H_d E_j + \frac{1}{2}\kappa_{ij}(\tau) L_i H_u L_j H_u$$

In diagonal Y_e basis:

- Modular invariant holomorphic observables

$$I_{ij}(\tau) = \frac{\mathscr{M}_{ii}(\tau)\mathscr{M}_{jj}(\tau)}{(\mathscr{M}_{ij}(\tau))^2} = \frac{\kappa_{ii}\,\kappa_{jj}}{\kappa_{ij}^2} = \frac{m_{ii}(\tau,\overline{\tau})m_{jj}(\tau,\overline{\tau})}{(m_{ij}(\tau,\overline{\tau}))^2}$$

I_{ij} invariant under renormalization group

Chang, Kuo (2002)

A Toy Modular A_4 Model

Feruglio (2017)

- Lepton sector of the (supersymmetric) standard model

	(E_1^c, E_2^c, E_3^c)	L	H_d	H_u	φ
$\mathrm{SU(2)_L \times U(1)_Y}$	$\mathbf{1}_1$	$\mathbf{2}_{-1/2}$	$\mathbf{2}_{-1/2}$	$\mathbf{2}_{1/2}$	$\mathbf{1}_0$
Γ_3	$(\mathbf{1}, \mathbf{1'}, \mathbf{1''})$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$
k	$(k_{E_1}, k_{E_2}, k_{E_3})$	k_L	k_d	k_u	k_φ

- Charged fermion masses obtained (with couplings to flavon) by adjusting 3 parameters
- Weinberg Operator: $\mathscr{W}_\nu = \frac{1}{\Lambda}\left[(H_u \cdot L)\, Y\, (H_u \cdot L)\right]_{\mathbf{1}}$
- Kähler potential of leptons: $K_L = \left(-\mathrm{i}\tau + \mathrm{i}\overline{\tau}\right)^{-1} \left(\overline{L}\, L\right)_{\mathbf{1}}$

Invariants in Toy Modular A_4 Model

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

- Mass matrix in canonical basis:

$$m_\nu(\tau,\bar\tau) = (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)\frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_2(\tau) & -Y_3(\tau) \\ -Y_2(\tau) & 2Y_3(\tau) & -Y_1(\tau) \\ -Y_3(\tau) & -Y_1(\tau) & 2Y_2(\tau) \end{pmatrix} =: (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)v_u^2\begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{pmatrix}$$

- Invariants

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\big(Y_2(\tau)\big)^2}, \qquad I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\big(Y_3(\tau)\big)^2}, \qquad I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\big(Y_1(\tau)\big)^2}$$

- Algebraic constraint $\qquad Y_2^2 + 2Y_1Y_3 = 0$

- Thus

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) = -2\Big(1+\frac{1}{3}j_3(\tau)\Big)^3, \qquad I_{23}(\tau) = -\frac{32}{I_{13}(\tau)}$$

Invariants in Toy Modular A_4 Model

- Two interesting relations: **RG invariant, independent of τ**

$$I_{12}(\tau) = -2, \qquad I_{13}(\tau) I_{23}(\tau) = -32$$

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

$\Rightarrow$ **sum rules among physical observables:**

RG invariant, τ independent

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A_4 Model

- Invariants I_{ij} : functions of physical observables

$$\left(m_1, m_2, m_3, \theta_{12}, \theta_{23}, \theta_{13}, \delta, \alpha_{12}, \alpha_{23}\right)$$

$$I_{12} = -2$$

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(e^{i\delta}c_{23}s_{12} + c_{12}s_{13}s_{23}\right)^2 + \widetilde{m}_2\left(e^{i\delta}c_{12}c_{23} - s_{12}s_{13}s_{23}\right)^2 + e^{2i\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(e^{i\delta}c_{23}s_{12} + c_{12}s_{13}s_{23}\right) + \widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23} - e^{i\delta}c_{12}c_{23}\right) - e^{2i\delta}m_3s_{13}s_{23}\right]^2}$$

$$\widetilde{m}_1 := m_1 e^{i\varphi_1}$$

$$\widetilde{m}_2 := m_2 e^{i\varphi_2}.$$

$$a_0 := \left(\widetilde{m}_1c_{12}^2 + \widetilde{m}_2s_{12}^2\right)c_{13}^2 + e^{2i\delta}m_3s_{13}^2 ,$$

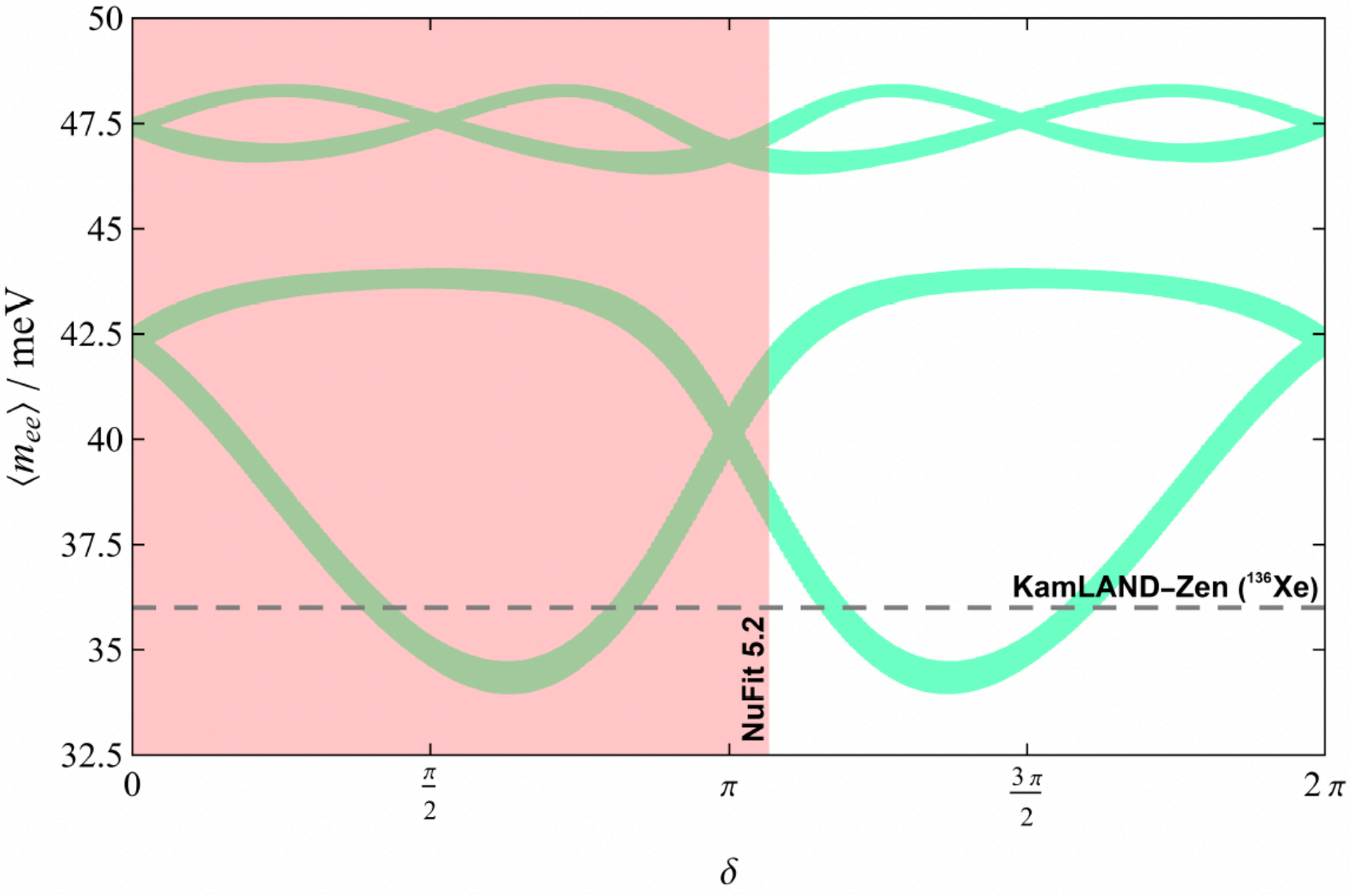
$$a_1 := \left[\left(e^{2i\delta}s_{12}^2 - c_{12}^2s_{13}^2\right)\sin(2\theta_{23}) - e^{i\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right] ,$$

$$a_2 := \left[e^{i\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13} + \left(e^{2i\delta}c_{12}^2 - s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right] .$$

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

Invariants in Toy Modular A_4 Model

- Predictions from $I_{12} = -2$ invariant, Inverted Ordering



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Reality of $I_{12} \Rightarrow$

Constraints on CP phases:

Given $\delta \Rightarrow$

Majorana phases α_{12}, α_{13}

Invariants in Toy Modular A_4 Model

- No simultaneous solution for all I_{ij} that is consistent with data
  - Agree with previous analysis by scanning parameter space (i.e. toy modular A4 model does not fit all data)
  - Here, arrived at conclusion without the need to scan

MCC, X.-Q. Li, X.-G. Liu, O. Medina, M. Ratz (2024)

- c.f. weak basis invariants $\rightarrow$ mass ordering, θ_{23} octant

G.C. Branco, M.N. Rebelo, J. I. Silva-Marcos (2017)

Validity at High Orders?

- **SUSY setups:**
  - RG corrections vanish to all orders
- **Non-SUSY setups: SM**
  - 2 loop:

$$\frac{\mathrm{d}I_{fg}}{\mathrm{d}t} = \frac{2(y_f^2 - y_g^2)^2}{(16\pi^2)^2} I_{fg}$$

MCC, S. Perez, M. Ratz (2026)

  - Corrections:

$$\Delta I_{fg} \approx \frac{2(y_f^2 - y_g^2)^2}{(16\pi^2)^2} I_{fg} \Delta t$$

  - Example: running between $10^3 - 10^6$ GeV
    - Negligible

$$\Delta I_{12} \approx -1.2 \times 10^{-15}$$
$$\Delta I_{13} \approx -1.3 \times 10^{-11}$$
$$\Delta I_{23} \approx 2.6 \times 10^{-12}$$

Determinants & VVMFs

- Determinants of mass matrices are 1-dim VVMFs

$$\det M(\tau) \xrightarrow{\gamma} (c\tau + d)^{\sum_i k_{\varphi_i} c + k_{\varphi_i}} \left(\det \rho^{\mathcal{C}} \rho\right)^* \det M(\tau)$$

X.-G. Liu, G.-J. Ding (2022)

- 1-dim VVMFs form a finite-dimensional linear space generated by η^2, E_4, E_6

$$\mathcal{M}_k^{(1D)}(\mathrm{SL}(2,\mathbb{Z})) = \sum_{\substack{a+b+c=k \\ a,b,c\geq 0}} \alpha_{abc}\, \eta^{2c}\, {E_4}^a\, {E_6}^b$$

η : Dedekind η function

$E_{4,\,6}$: Eisenstein series

- We know a lot about the determinants from theory of modular forms

Location of Zeros

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

c.f. Feruglio, Gherardi, Romanino, Titov (2021); Novichkov, Penedo, Petcov (2021); Feruglio (2023)

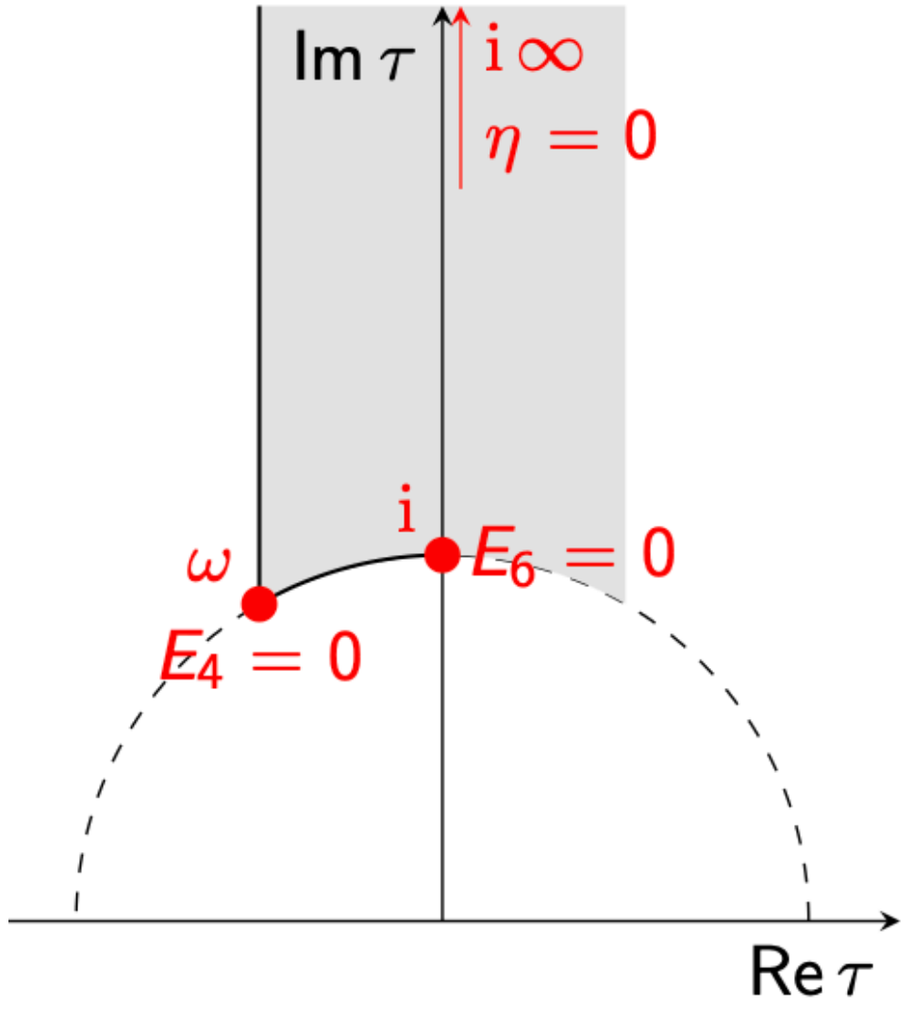
- Close to zeros of det M $\Rightarrow$ hierarchical masses

- Orders of zeros constrained by Valence Formula

Prop 2 in Bruinier, van der Geer, Harder, Zagier (2008)

$$\frac{1}{12}n_{\mathrm{i}\,\infty}+\frac{1}{2}n_{\mathrm{i}}+\frac{1}{3}n_{\omega}+\sum_{\substack{\tau_0\neq\\ \mathrm{i},\omega,\mathrm{i}\,\infty}} n_{\tau_0}=\frac{k}{12}$$

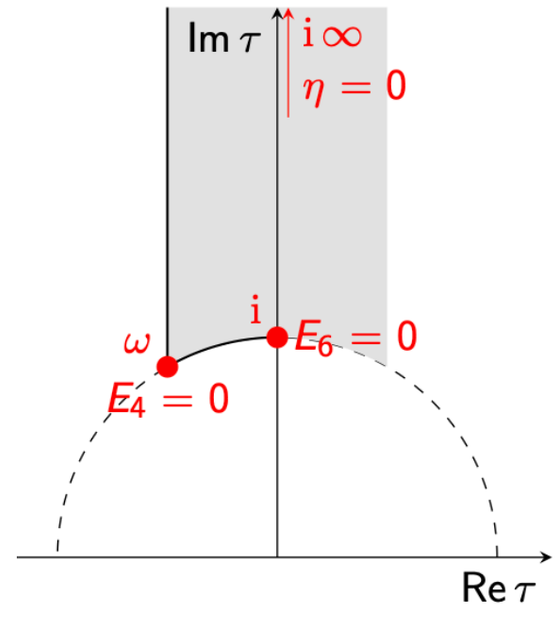
- Absence of cancellations of different terms for $k_{\det M}<12$, the zeros can only be at $\tau=\omega,\, i,\, i\infty$



Determinants of Weights up to 6

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\infty}, n_{\mathrm{i}}, n_\omega)$	irrep
1	η^2	$\mathrm{i}\infty$	$(1,0,0)$	$\mathbf{1}_1$
2	η^4	$\mathrm{i}\infty$	$(2,0,0)$	$\mathbf{1}_2$
3	η^6	$\mathrm{i}\infty$	$(3,0,0)$	$\mathbf{1}_3$
4	E_4	ω	$(0,0,1)$	$\mathbf{1}_0$
	η^8	$\mathrm{i}\infty$	$(4,0,0)$	$\mathbf{1}_4$
5	$\eta^2 E_4$	ω and $\mathrm{i}\infty$	$(1,0,1)$	$\mathbf{1}_1$
	η^{10}	$\mathrm{i}\infty$	$(5,0,0)$	$\mathbf{1}_5$
6	E_6	i	$(0,1,0)$	$\mathbf{1}_0$
	$\eta^4 E_4$	ω and $\mathrm{i}\infty$	$(2,0,1)$	$\mathbf{1}_2$
	η^{12}	$\mathrm{i}\infty$	$(6,0,0)$	$\mathbf{1}_6$



$$\det M(\tau) \xmapsto{\gamma}$$

$$(c\tau + d)^{\sum_i k_{\varphi_i} c + k_{\varphi_i}} \left(\det \rho^{\mathcal{C}} \rho\right)^* \det M(\tau)$$

$$\frac{1}{12} n_{\mathrm{i}\infty} + \frac{1}{2} n_{\mathrm{i}} + \frac{1}{3} n_\omega + \sum_{\substack{\tau_0 \neq \\ \mathrm{i}, \omega, \mathrm{i}\infty}} n_{\tau_0} = \frac{k}{12}$$

Orders of Zeros and Mass Hierarchy

MCC, X.-Q. Li, X.-G. Liu, M. Ratz (2025)

- Convincing explanation of hierarchical masses by modular flavor symmetries $\Rightarrow$ requires zeros of high orders

- At $\tau = i$ or ω : zeros at most orders 1 or 2

- At $\tau = i\infty$: only occur for nontrivial 1-plets: $1_{p>0}$

- Certain finite modular groups do not have appropriate $1_{p>0}$-plets

- Perfect groups have no nontrivial 1-plets, e.g. MCC, Fallbacher, Ratz, Trautner, Vaudrevange (2015)
  - A_5 : hierarchy can only be obtained by balancing coefficients

- Bottom line: given finite modular group + modular weights of matter fields $\Rightarrow$ can immediately tell whether mass hierarchy can emerge as properties of modular forms

Probing Flavor Symmetries?

- Many fermion mass models based on non-Abelian discrete flavor symmetries or modular flavor symmetries

  - **Signatures at particle physics experiments**
    - Correlations among physical observables
    - Symmetry breaking scale? Degeneracy?

  See talks by Claudia Hagedorn, Ana Teixeira

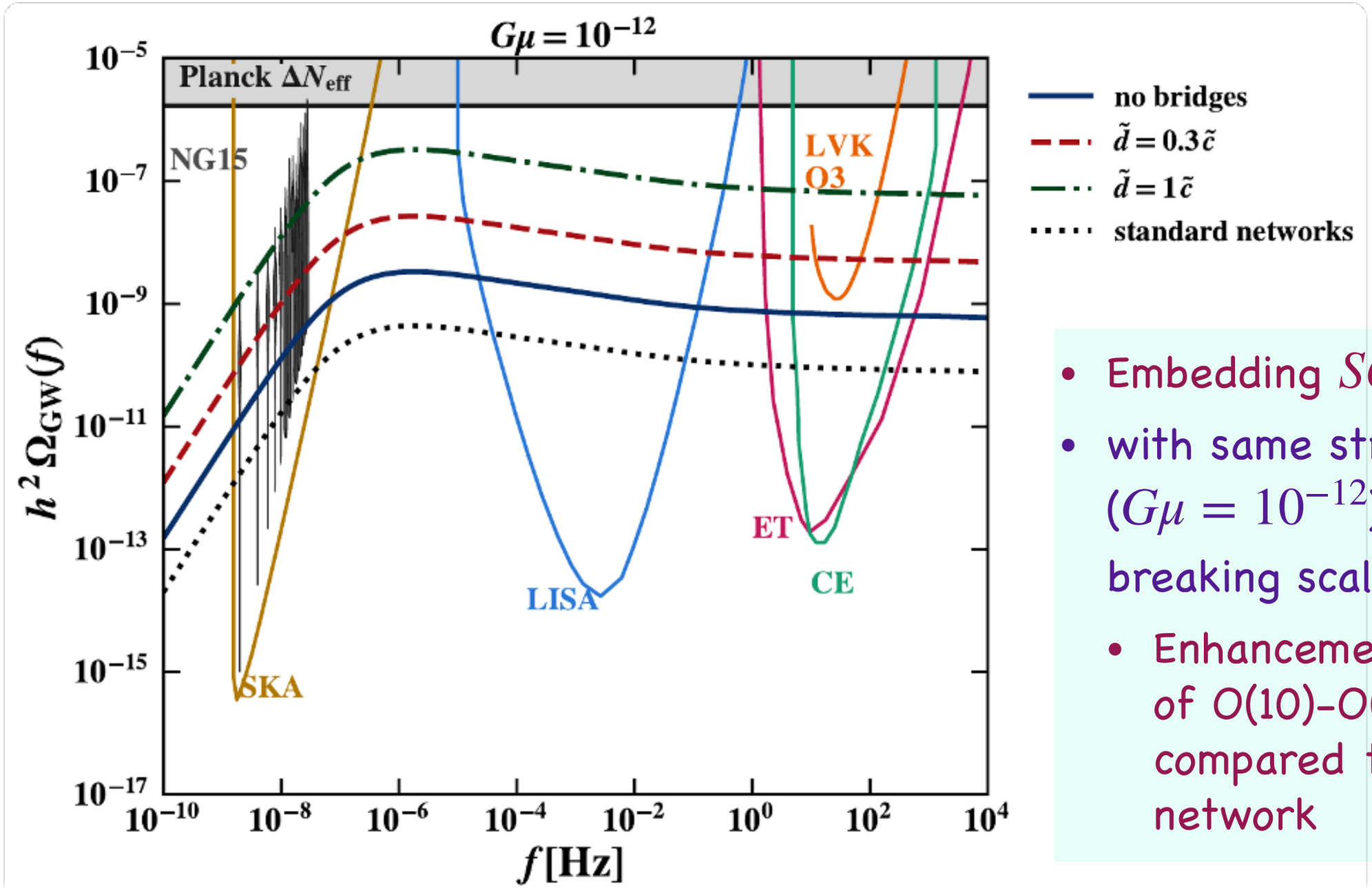
- **Topological defects**
  - Domain walls collapse: observable GW signature
    - A_4: cross flavon coupling consistent w/ symmetry ⇒ $\theta_{13} \neq 0$, bias term ⇒ realistic flavor model, observable GW spectrum

  see talks by Jessica Turner, Arsenii Titov, Miguel Levy

MCC, Harold Matias, Cameron Moffett-Smith (2026)

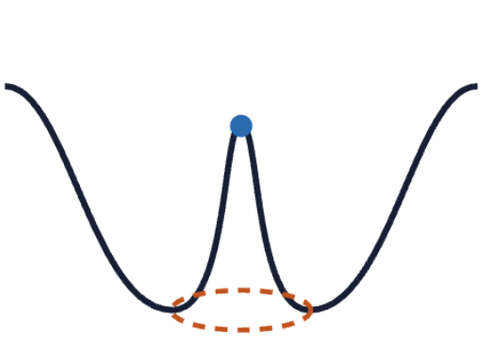
GW Signals from Cosmic Strings

MCC, Cameron Moffett-Smith (2026)

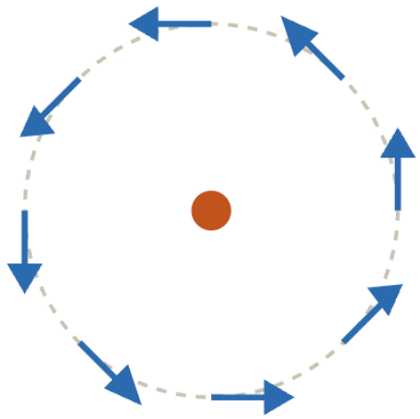


- Embedding $SO(3) \to A_4$
- with same string tension ($G\mu = 10^{-12}$), symmetry breaking scale (10^{13} GeV):
  - Enhancement by a factor of O(10)-O(1000) compared to standard U(1) network

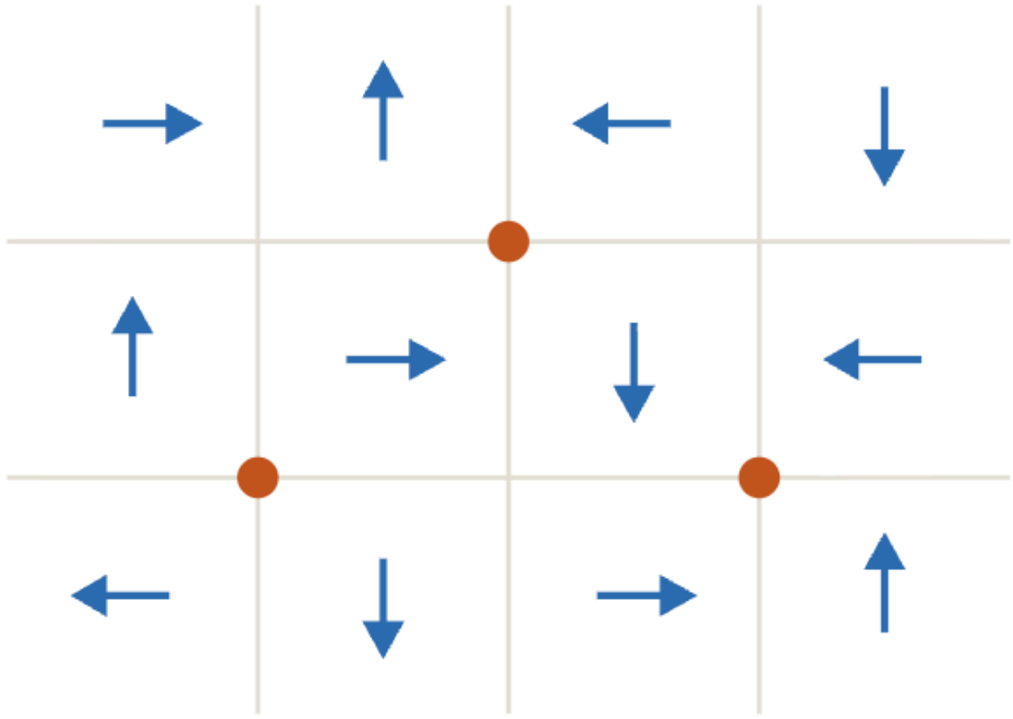
Cosmic Strings



Vacua: a circle of phases θ



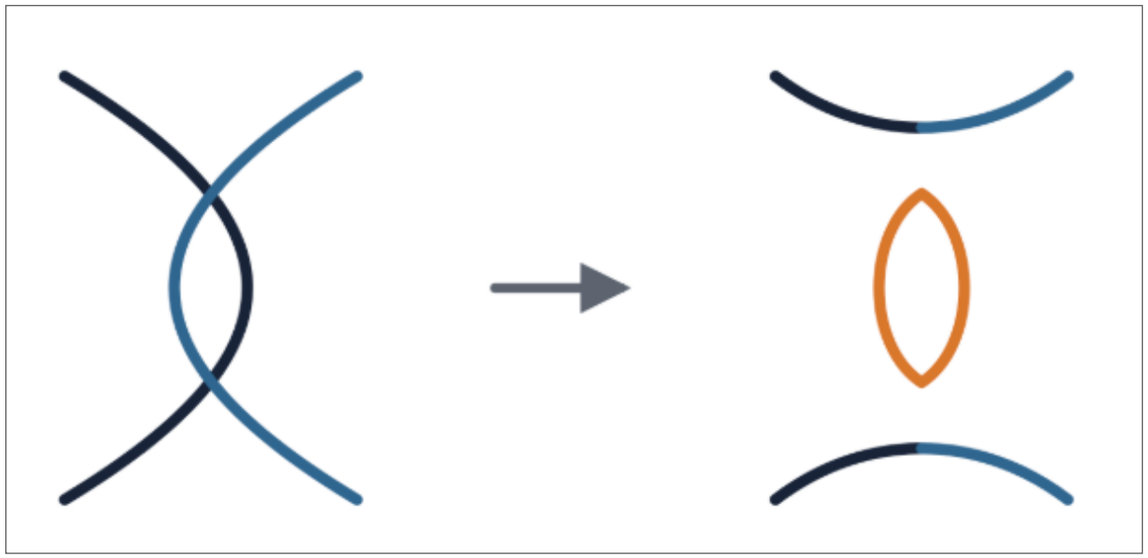
θ winds once: string at the centre



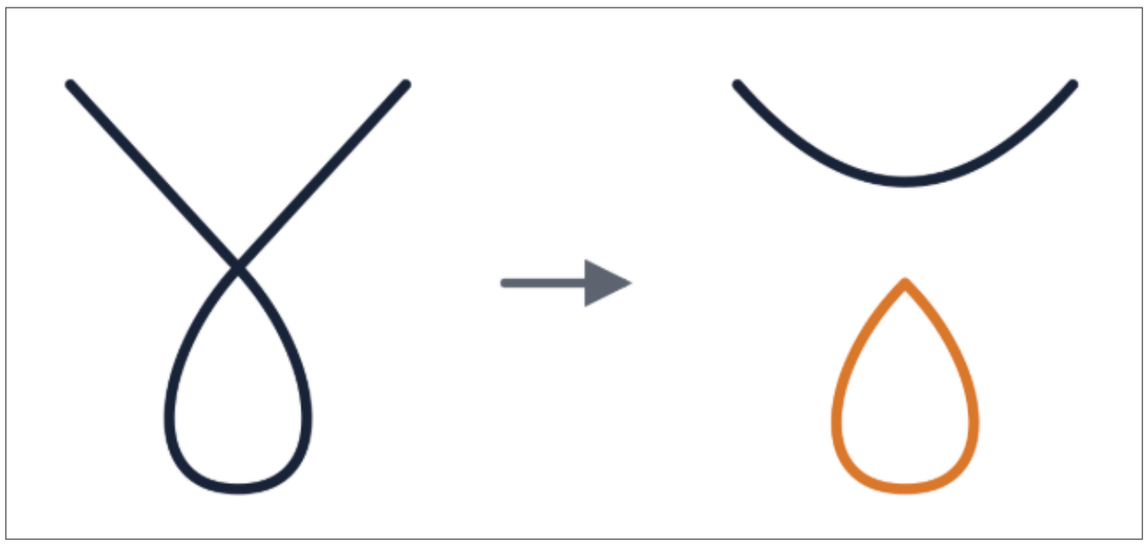
- Causally disconnected domains select uncorrelated vacuum phases
- Where the phase winds by 2π around a domain junction ⇒ a string forms (orange dots)

Kibble (1976); Zurek (1985)

Inter-commuting & Loop Production: Abelian strings



Two strings inter-commuting: a loop is chopped off



A string inter-commuting with itself: a loop is chopped

Gravitational Wave Production

- Each loop oscillates and releases energy through GW until it disappears
- Summing all loops over cosmic history ⇒ stochastic GW spectrum
- Three factors:

String tension

$G\mu \leftrightarrow$ symmetry breaking scale

Spectrum plateau height $\sim (G\mu)^{1/2}$

Loop size and birth time

$\ell = \alpha t \leftrightarrow$ frequency

Earlier, smaller loops radiate at high frequencies today

Number of loops

$$\frac{dn}{dt} \propto \frac{1}{\gamma^3} \quad \leftrightarrow \quad \text{amplitude}$$

A denser (smaller γ) network makes more loops, producing stronger signal

Velocity-dependent one-scale (VOS) model: Martins, Shellard (1996, 2002)

Gouttenoire, Servant, Simakachorn (2020); Blanco-Pillado, Olum, Shlaer (2014)

String Collision Outcomes: Non-Abelian Strings

Two additional outcomes possible: Pass Through, Bridge ⇒ no loops

Probabilities determined by group theory

MCC, Cameron Moffett-Smith (2026)

Reconnect



b = a or a^{-1}

Pass through

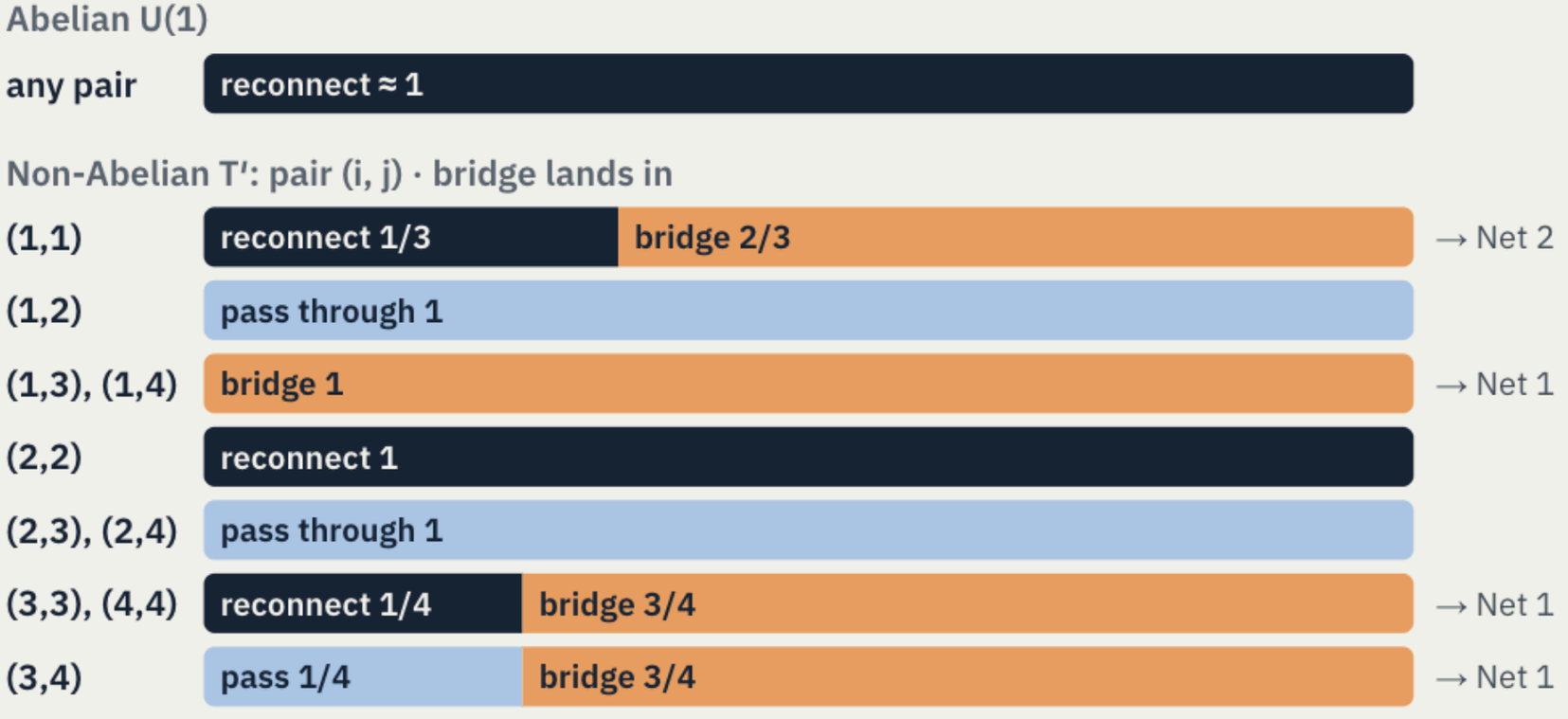


ab = ba, but b ≠ a, a^{-1}

Bridge

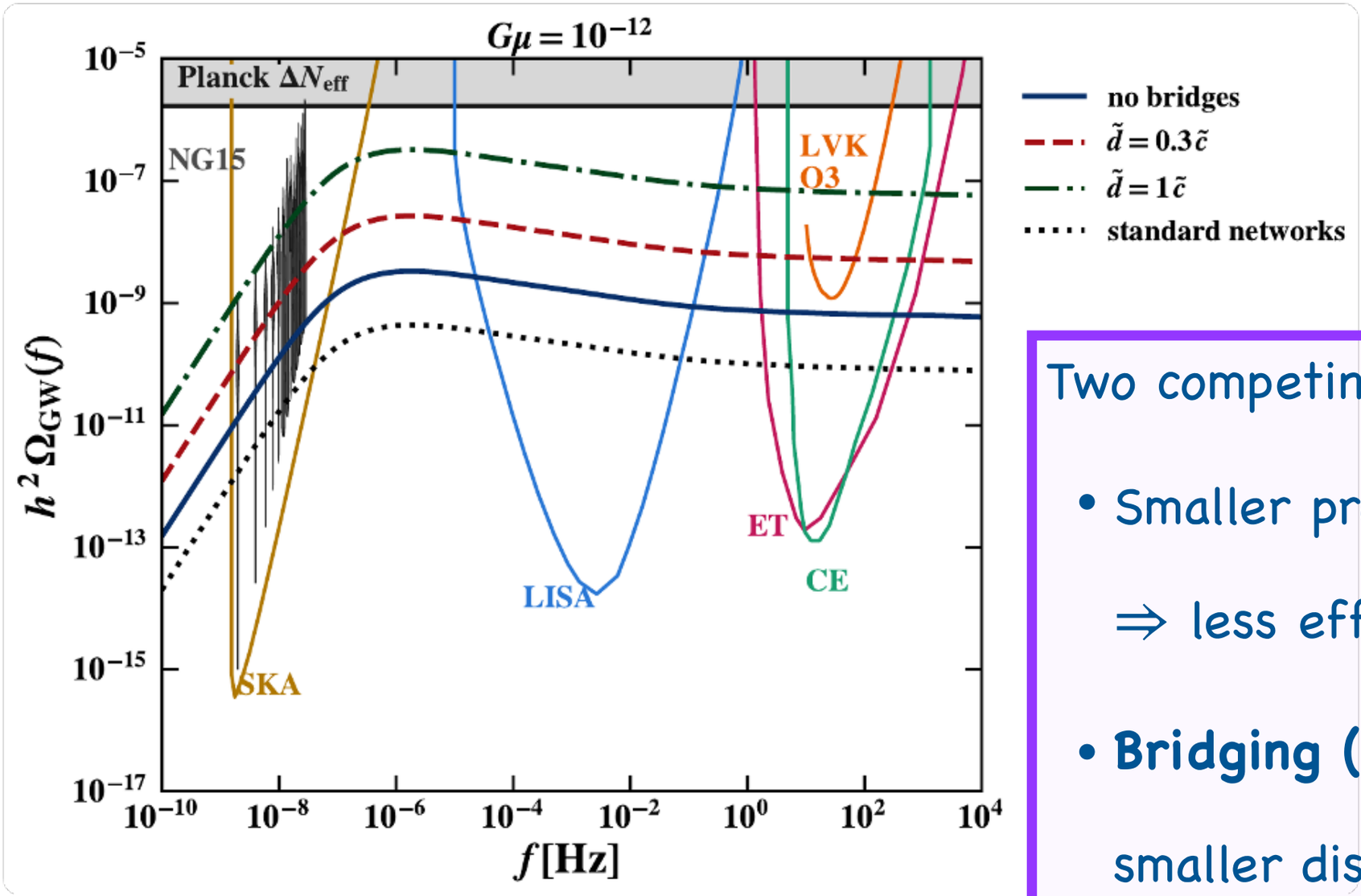


ab ≠ ba: new string with flux $aba^{-1}b^{-1}$



Enhancement of GW signals in Non-Abelian Network

MCC, Cameron Moffett-Smith (2026)



Two competing effects

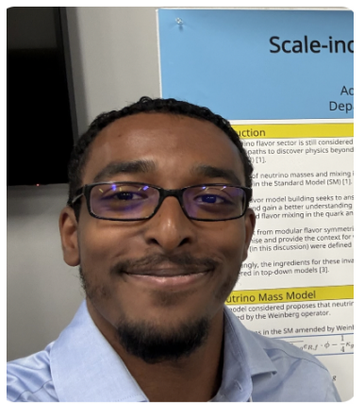
- Smaller probability for **reconnection** ⇒ less efficient loop production
- **Bridging (= adding strings)** ⇒ smaller distance between strings ⇒ denser network ⇒ enhanced loop production

Velocity-dependent one-scale (VOS) model:
Martins, Shellard (1996, 2002)

Acknowledgements



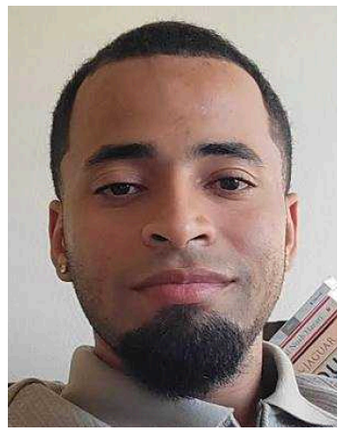
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MCC, S. Perez, M. Ratz, Universe 12 (2026) 2, 46

Conclusions

- Modular Flavor Symmetries:
  - Significant reduction of the number of parameters
  - Modular Invariant (and even τ-independent), RG Invariants:
    - Robust sum rules among physical observables
    - Near-critical behavior of mass matrices:
      - Mass hierarchy: deviation from fixed points
      - Classification of symmetries based on modular weights
      - Preference for $\tau = i\infty$
- Gravitational Waves: probing underlying flavor symmetry
  - Enhancement in Non-Abelian cosmic string network
  - Quantitative prediction requires simulations for NA network

To appear in arXiv on Tuesday

- In any supersymmetric theory also describing gravity, modular symmetries are R-symmetries
  - Important restrictions when trying to combine modular symmetries with 4D GUTs

MCC, Xiang-Gan Liu, Michael Ratz, Alex Stewart (2026)



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Renormalization Group Invariants

- In P-diagonal basis: $P = C_e Y_e^\dagger Y_e$ Chang, Kuo (2002)

$$\frac{d}{dt}\kappa = \tilde{P}\kappa\tilde{Q}^T + \tilde{Q}\kappa\tilde{P}^T + \tilde{\alpha}\kappa,$$

$$\tilde{P} = \text{diag}\left(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3\right), \quad \tilde{Q} = \text{diag}\left(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3\right)$$

- At 1-loop: $\tilde{P} = \frac{1}{16\pi^2}P, \quad \tilde{Q} = I, \quad \tilde{\alpha} = \frac{1}{16\pi^2}\alpha$

$$\dot{\kappa}_{ij} = \kappa_{ij}\left(\tilde{P}_i\tilde{Q}_j + \tilde{P}_j\tilde{Q}_i + \tilde{\alpha}\right), \quad \frac{d}{dt}I_{ij} = 2\left(\tilde{P}_i - \tilde{P}_j\right)\left(\tilde{Q}_i - \tilde{Q}_j\right)I_{ij}$$

$\Rightarrow I_{ij}$ is RG invariant

Invariants in Toy Modular A_4 Model

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+\mathrm{e}^{2\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}\right)-\mathrm{e}^{2\mathrm{i}\,\delta}m_3s_{13}s_{23}\right]^2}\,,$$

$$I_{13} = \frac{a_0\left[\widetilde{m}_1\left(c_{12}c_{23}s_{13}-\mathrm{e}^{\mathrm{i}\,\delta}s_{12}s_{23}\right)^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+\mathrm{e}^{\mathrm{i}\,\delta}c_{12}s_{23}\right)^2+\mathrm{e}^{2\mathrm{i}\,\delta}m_3c_{13}^2c_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(c_{12}c_{23}s_{13}-\mathrm{e}^{\mathrm{i}\,\delta}s_{12}s_{23}\right)+\widetilde{m}_2s_{12}\left(c_{23}s_{12}s_{13}+\mathrm{e}^{\mathrm{i}\,\delta}c_{12}s_{23}\right)-\mathrm{e}^{2\mathrm{i}\,\delta}m_3c_{23}s_{13}\right]^2}\,,$$

$$I_{23} = \left[\mathrm{e}^{2\mathrm{i}\,\delta}m_3c_{13}^2s_{23}^2+\widetilde{m}_1\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(\mathrm{e}^{\mathrm{i}\,\delta}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2\right]$$
$$\times\frac{4\left[\mathrm{e}^{2\mathrm{i}\,\delta}m_3c_{13}^2c_{23}^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+\mathrm{e}^{\mathrm{i}\,\delta}c_{12}s_{23}\right)^2+\widetilde{m}_1\left(c_{12}c_{23}s_{13}-\mathrm{e}^{\mathrm{i}\,\delta}s_{12}s_{23}\right)^2\right]}{\left[\widetilde{m}_1a_1+\widetilde{m}_2a_2-\mathrm{e}^{2\mathrm{i}\,\delta}m_3\sin(2\theta_{23})c_{13}^2\right]^2}\,,$$

$$\widetilde{m}_1 := m_1\,\mathrm{e}^{\mathrm{i}\,\varphi_1}$$

$$\widetilde{m}_2 := m_2\,\mathrm{e}^{\mathrm{i}\,\varphi_2}.$$

$$a_0 := \left(\widetilde{m}_1c_{12}^2+\widetilde{m}_2s_{12}^2\right)c_{13}^2+\mathrm{e}^{2\mathrm{i}\,\delta}m_3s_{13}^2\,,$$

$$a_1 := \left[\left(\mathrm{e}^{2\mathrm{i}\,\delta}s_{12}^2-c_{12}^2s_{13}^2\right)\sin(2\theta_{23})-\mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right]\,,$$

$$a_2 := \left[\mathrm{e}^{\mathrm{i}\,\delta}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}+\left(\mathrm{e}^{2\mathrm{i}\,\delta}c_{12}^2-s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right]\,.$$

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$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\infty}, n_{\mathrm{i}}, n_{\omega})$	irrep
7	$\eta^6 E_4$	ω and $\mathrm{i}\infty$	$(3,0,1)$	$\mathbf{1}_3$
	$\eta^2 E_6$	i and $\mathrm{i}\infty$	$(1,1,0)$	$\mathbf{1}_1$
	η^{14}	$\mathrm{i}\infty$	$(7,0,0)$	$\mathbf{1}_7$
8	E_4^2	ω	$(0,0,2)$	$\mathbf{1}_0$
	$\eta^4 E_6$	i and $\mathrm{i}\infty$	$(2,1,0)$	$\mathbf{1}_2$
	$\eta^8 E_4$	ω and $\mathrm{i}\infty$	$(4,0,1)$	$\mathbf{1}_4$
	η^{16}	$\mathrm{i}\infty$	$(8,0,0)$	$\mathbf{1}_8$
9	$\eta^{10} E_4$	ω and $\mathrm{i}\infty$	$(5,0,1)$	$\mathbf{1}_5$
	$\eta^6 E_6$	i and $\mathrm{i}\infty$	$(3,1,0)$	$\mathbf{1}_3$
	$\eta^2 E_4^2$	ω and $\mathrm{i}\infty$	$(1,0,2)$	$\mathbf{1}_1$
	η^{18}	$\mathrm{i}\infty$	$(9,0,0)$	$\mathbf{1}_9$

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$k_{\det M}$	Modular Form	Zeros	$(n_{\mathrm{i}\infty}, n_{\mathrm{i}}, n_{\omega})$	irrep
10	E_4E_6	ω and i	$(0,1,1)$	$\mathbf{1}_0$
	$\eta^4E_4^2$	ω and $\mathrm{i}\infty$	$(2,0,2)$	$\mathbf{1}_2$
	η^8E_6	i and $\mathrm{i}\infty$	$(4,1,0)$	$\mathbf{1}_4$
	$\eta^{12}E_4$	ω and $\mathrm{i}\infty$	$(6,0,1)$	$\mathbf{1}_6$
	η^{20}	$\mathrm{i}\infty$	$(10,0,0)$	$\mathbf{1}_{10}$
11	$\eta^2E_4E_6$	ω, i and $\mathrm{i}\infty$	$(1,1,1)$	$\mathbf{1}_1$
	$\eta^6E_4^2$	ω and $\mathrm{i}\infty$	$(3,0,2)$	$\mathbf{1}_3$
	$\eta^{10}E_6$	i and $\mathrm{i}\infty$	$(5,1,0)$	$\mathbf{1}_5$
	$\eta^{14}E_4$	ω and $\mathrm{i}\infty$	$(7,0,1)$	$\mathbf{1}_7$
	η^{22}	$\mathrm{i}\infty$	$(11,0,0)$	$\mathbf{1}_{11}$