CP CONSERVATION IN THE STRONG INTERACTIONS

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Outline

- Functional quantization
- Canonical quantization and θ-vacua
- Effective Lagrangians, $U(1)_A$

[2001.07152, 2403.00747, 2404.16026, 2511.04216]

FUNCTIONAL QUANTIZATION

Topological term

$$\mathcal{L} \supset \frac{1}{2g^2}\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} - \frac{\mathrm{i}}{16\pi^2}\theta\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu} \qquad \tilde{F}_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

The topological term is parity-odd and a total divergence:

gauge invariant

$$\frac{1}{4}\mathrm{tr}\,F_{\mu\nu}\tilde{F}_{\mu\nu} = \partial_\mu K_\mu$$

$$K_\mu = \epsilon_{\mu\nu\alpha\beta}\mathrm{tr}\left[\frac{1}{2}A_\nu\partial_\alpha A_\beta + \frac{1}{3}A_\nu A_\alpha A_\beta\right]$$

gauge dependent

→ No perturbation theory vertices → Unlike other Lagrangian terms

→ Topological properties through surface integrals

Topological arguments in Yang–Mills theory either rely on

A vacuum boundary conditions on four dimensional spacetimes (gauge independent)

B or on point compactification of three-dimensional space, imposing a gauge-dependent restriction of allowed gauge transformations

A Topology in four-dimensional spacetime

$$U = \begin{pmatrix} a_R + i a_I & -b_R + i b_I \\ b_R + i b_I & a_R - i a_I \end{pmatrix} \in \mathrm{SU}(2) \text{ for } a_R^2 + a_I^2 + b_R^2 + b_I^2 = 1$$

$\Rightarrow$Homotopy: $\mathrm{SU}(3) \supset \mathrm{SU}(2) \cong S^3 \longrightarrow \pi_3(\mathrm{SU}(2)) = \pi_3(S^3) = \mathbb{Z}$

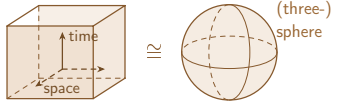
Topological quantization for pure gauge $A_\mu \to -\frac{i}{g}(\partial_\mu U)U^{-1}$ at $\partial\Omega \cong S^3$

$$\Delta n = \frac{1}{16\pi^2}\int_\Omega \mathrm{d}^4x F_{\mu\nu}\tilde{F}_{\mu\nu} = \frac{1}{4\pi^2}\oint_{\partial\Omega} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

[see e.g. Coleman, "Aspects of symmetry" (1985)]

Haar measure for pure gauge
$K_\mu = \frac{1}{6}\varepsilon_{\mu\nu\lambda\rho}\mathrm{tr}[(U^{-1}\partial_\nu U)(U^{-1}\partial_\lambda U)(U^{-1}\partial_\rho U)]$

E.g. take boundary of $\Omega = \mathbb{R}^4$ as a sphere S^3:



Or $\Omega = T^4$(lattice), $\Omega = S^4$(Euclidean dS): $\Delta n \in \mathbb{Z}$ based on slightly more involved argument

The value of Δn as well as the fact that $\Delta n \in \mathbb{Z}$ are *gauge independent.*

B Topology on spatial hypersurfaces—point compactification, large gauge transformations

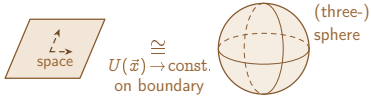
Consider temporal gauge $A^0 = 0$ (in view of canonical quantization)

Chern–Simons functional:

$$W[\vec{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_V \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_iA_jA_k\right] \equiv \frac{1}{4\pi^2}\int_V \mathrm{d}^3x\,K_0$$

Define $\vec{A}_U = U\vec{A}U^{-1} + \mathrm{i}U^{-1}\vec{\nabla}U$ (residual gauge freedom in temporal gauge)

With extra condition $U(\vec{x}) \to$ const. on ∂V (periodic on T^3)
$\to$ Point compactification, homotopy $V \cong S^3$ ($V \cong T^3$)



$U^{(\nu)}$: equivalence classes of "large" ($\nu \neq 0$) gauge transformation on spacelike ($t =$ const.) hypersurface $V \simeq S^3$ ($V \simeq T^3$) with $W[\vec{A}_{U^{(\nu)}}] - W[\vec{A}] = \nu \in \mathbb{Z}$

However: Construction relies on *gauge choice* $U(\vec{x}) \to$ const. on ∂V (periodic on T^3)

Why $T \to \infty$ (implying $\mathrm{Vol}(\Omega) = VT \to \infty$)

Observables from pure gauge bcs on finite $\partial\Omega$ are not ground state $|0\rangle$ expectation values

For finite T amplitude, must project path integral on wave functional $\Psi_0[\phi(\vec{x})] = \langle \phi(\vec{x})|0\rangle$:

$$\langle 0, t_f|0, t_i\rangle = \int \mathcal{D}\phi_f \mathcal{D}\phi_i \langle 0|\phi_f\rangle \int_{\substack{\phi(t_f,\vec{x})=\phi_f(\vec{x}) \\ \phi(t_i,\vec{x})=\phi_i(\vec{x})}} \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}S[\phi]} \langle \phi_i|0\rangle \qquad T = t_f - t_i$$

$|\phi\rangle$ field eigenstates, not energy eigenstates

Problem: Don't know $\Psi_0[\phi(\vec{x})]$

Way out: Euclidean path integral/project on ground state

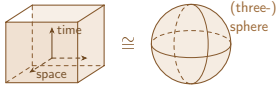
$$\lim_{T\to\infty} \frac{\mathrm{e}^{-HT}}{\mathrm{e}^{-E_0 T}} \quad \text{or} \quad \lim_{T\to\infty} \frac{\mathrm{e}^{-\mathrm{i}HT(1-\mathrm{i}\varepsilon)}}{\mathrm{e}^{-\mathrm{i}E_0 T(1-\mathrm{i}\varepsilon)}} \qquad \begin{aligned} H&: \text{ Hamiltonian} \\ E_0&: \text{ ground state energy} \end{aligned}$$

$\to$ Obtain vacuum correlations without bothering about $\Psi_0[\phi(\vec{x})]$

$\Delta n \in \mathbb{Z}$ and order of limits

Finite action S for $\mathrm{Vol}(\Omega) = VT \to \infty \Rightarrow A_\mu \to -\frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}$ at infinity

For $\Omega = \mathbb{R}^4$ (infinite), $\partial\Omega \cong S^3$:



$$VT \to \infty \Rightarrow \Delta n \in \mathbb{Z}$$

Primary order of limits:

$$Z = \lim_{N\to\infty} \sum_{\Delta n=-N}^{N} \lim_{VT\to\infty} \underbrace{\int_{\Delta n} \mathcal{D}A\, \mathrm{e}^{-S_{\mathrm{E}}[A]}}_{=:Z_{\Delta n}}$$

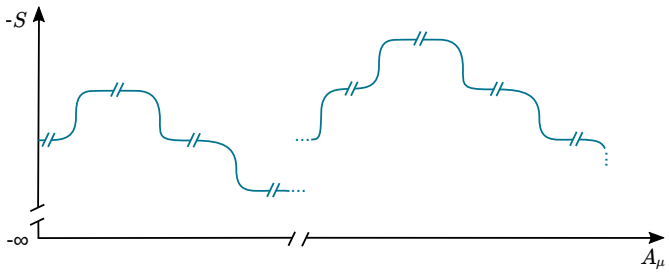
Only difference with conventional approach

More technically: Integration contour from Lefschetz thimbles

Parametrization of the path integral through steepest descent contours about classical saddle points $\longrightarrow$ Contour integration on Lefschetz thimbles

$$\frac{\partial\phi(x;u)}{\partial u} = \overline{\frac{\delta S_{\mathrm{E}}[\phi(x;u)]}{\delta\phi(x;u)}} \Longrightarrow -\frac{\partial \mathrm{Re} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} \leq 0 \text{ and } \frac{\partial \mathrm{Im} S_{\mathrm{E}}[\phi(x;u)]}{\partial u} = 0$$

Each thimble emerges from a critical point and corresponds to one $\Delta n \in \mathbb{Z}$



Keeping VT finite while summing over different Δn ($\Leftrightarrow$ different boundary conditions, infinite distance in field space) does *not* correspond to a nonsingular deformation of the Cauchy contour

Integration contour sweeps over full thimbles first, i.e. $VT \to \infty$ before sum over Δn

- Take $\langle F(x)\tilde{F}(x)\rangle$ as measure for CP violation
- Each element in the sequence over N vanishes:

$$\langle F(x)\tilde{F}(x)\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \frac{\Delta n}{VT} Z_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}} = 0$$

- When limits are *exchanged* the other way around: $\langle F(x)\tilde{F}(x)\rangle = \mathcal{O}(\theta)$
- Order of limits matters because series is not positive definite due to phases $e^{i\Delta n\theta}$, not absolutely summable

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CP conserved
in the path integral approach

CANONICAL QUANTIZATION

Theta vacuum, standard story

Take, $A^0 = 0$, *assume* in addition:

For $|\vec{x}| \to \infty$: $\vec{A}(\vec{x}) = \mathrm{i} U^{-1}(\vec{x}) \vec{\nabla} U(\vec{x})$ and $U(\vec{x}) \to \text{const.}$

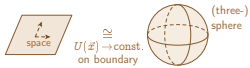
But why?

Consider initial and final states, taking $x_4 \to \pm\infty$

$\to$ *Ansatz*: Construct from pure gauge configurations on these surfaces, with

$$n_{\pm\infty} = \frac{1}{4\pi^2} \int\limits_{x^4 = \pm\infty} \mathrm{d}^3\sigma K_\perp \in \mathbb{Z}$$

point compactification



Prevacua: $n_{-\infty} \to |n\rangle$ (field eigenstates)
$n_{\infty} \to \langle n|$

Gauge invariant (up to phase) state $|\theta\rangle = \sum\limits_n \mathrm{e}^{-\mathrm{i}n\theta} |n\rangle$ [Callan, Dashen, Gross (1976); Jackiw, Rebbi (1976); Jackiw (1980)]

⚠ States not normalizable in the proper sense: $\langle \theta^{(i)} | \theta^{(j)} \rangle = \sum\limits_n \delta(\theta^{(i)} - \theta^{(j)} + 2\pi n)$

[cf. e.g. Callan, Dashen, Gross (1976); issue taken by Okubo, Marshak (1992)]

Without ado, this contradicts 1st Dirac–von Neumann axiom of quantum mechanics.

[Dirac 1932, von Neumann 1932]

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n_{∞} —

B. STATEMENT OF THE POSTULATES

1. Description of the state of a system

In chapter I, we introduced the concept of the quantum state of a particle. We first characterized this state at a given time by a square-integrable wave function. Then, in chapter II, we associated a ket of the state space $\mathscr{E}_r$ with each wave function: choosing $|\psi\rangle$ belonging to $\mathscr{E}_r$ is equivalent to choosing the corresponding function $\psi(\mathbf{r}) = \langle \mathbf{r} | \psi \rangle$. Therefore, the quantum state of a particle at a fixed time is characterized by a ket of the space $\mathscr{E}_r$. In this form, the concept of a state can be generalized to any physical system.

First Postulate: At a fixed time t_0, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to the state space $\mathscr{E}$.

[Cohen-Tanoudji, Diu, Laloë]

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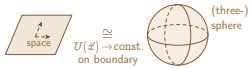
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Canonical quantization of the gauge field

Minkowski spacetime, temporal gauge $A^0 = 0$, no sources $\longrightarrow$

$$g\vec{E}^a = -\partial/\partial t\, \vec{A}^a$$

$$g\vec{B}^a = \vec{\nabla} \times \vec{A}^a - 1/2\, f^{abc}\vec{A}^a \times \vec{A}^b$$

Canonical momentum conjugate to $\vec{A}^a$:

$$g\vec{\Pi}^a = -\vec{E}^a + \frac{g^2}{8\pi^2}\theta\vec{B}^a$$

- **No gauge conditions on ∂V accounted for $\longrightarrow$ $\Psi[\vec{A}]$ must be defined for $U(\vec{x}) \neq \text{const.}$ on ∂V**
- **Residual gauge dofs.: Throw out unphysical states "First quantize, then constrain"**

The corresponding operator must observe the commutation relations:

$$[A^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\vec{x} - \vec{x}\,'), \; [\Pi^{a,i}(\vec{x}), \Pi^{b,j}(\vec{x}\,')] = 0, \quad \text{e.g. } \vec{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\vec{A}^a} + \theta_\Pi\frac{g^2}{8\pi^2}\vec{B}^a$$

Hamiltonian density:

$$\mathcal{H} = \frac{1}{2}\left((\vec{E}^a)^2 + (\vec{B}^a)^2\right) = \frac{1}{2}\left(\left(g\frac{\delta}{\mathrm{i}\delta\vec{A}^a} - \frac{g^2}{8\pi^2}(\theta - \theta_\Pi)\vec{B}^a\right)^2 + (\vec{B}^a)^2\right)$$

[e.g. Jackiw (1980)]

Since $[U^{(\nu)}, H] = 0$, can find states $\Psi_{\theta^{(i)}}[\vec{A}_{(U^{(1)})^\nu}] = \mathrm{e}^{\mathrm{i}\nu\theta^{(i)}}\Psi_{\theta^{(i)}}[\vec{A}]$

Is there a good reason for restricting $U(\vec{x})$?

$$U(\vec{x}) \xrightarrow[|\vec{x}|\to\infty]{} \text{const. not enforced by } A^0 = 0 \text{ only}$$

Only difference here with standard approach:
Carry out canonical quantization, constrain states for all gauges **without** the extra condition

To carry out this analysis it is convenient to impose the gauge condition $A_0 = 0$. Classically the ground state must correspond to time-independent field configurations of vanishing energy density.

We have therefore $F_{\mu\nu} \equiv 0$, which means that the field $\mathbf{A}$ is a pure gauge

$$\mathbf{A}(\mathbf{x}) = \partial[g(\mathbf{x})]g^{-1}(\mathbf{x}) \tag{12-33}$$

Furthermore, we assume that we may restrict ourselves to transformation functions $g(\mathbf{x})$ that have the same limit in all spatial directions. We may take this limit to be the identity in the group

$$g(\mathbf{x}) \xrightarrow[|\mathbf{x}|\to\infty]{} e \tag{12-34}$$

(there is actually no very convincing argument to justify this restriction). Under these circumstances, all the field configurations of the form (12-33) and (12-34) may be regarded as describing a ground state. We may wonder whether all these copies of the vacuum are equivalent, i.e., whether there exists

[Itzykson and Zuber, *Quantum Field Theory* (1980)]

In this section, I shall remain with the formalism as developed thus far, and study further the action of the gauge symmetry. We have remarked already on the invariance of the quantized theory, when $A^0 = 0$, under transformations which in infinitesimal form are described by Eq. (31), and in finite form by

$$\mathbf{A} \to U^{-1}\mathbf{A}U - \frac{1}{g}U^{-1}\nabla U \,. \tag{36a}$$

Here U is a 2×2 unitary, c-number SU(2) matrix, depending on position, but not on time. We shall make a very important hypothesis concerning the physically admissible finite transformations. While some plausible arguments can be given in support of this hypothesis (see below) in the end we must recognize it as an assumption, without which the subsequent development cannot be made. We shall assume that the allowed gauge transformation matrices U tend to a definite limit as r passes to infinity.

$$\lim_{r\to\infty} U(\mathbf{r}) = U_\infty \,. \tag{36b}$$

[Jackiw, *Introduction to the Yang–Mills quantum theory* (1980)]

Gauge fixed inner product

Include now *all* $U(\vec{x})$

Normalization issue readily addressed by *gauge fixing*:
Integrate over $\vec{A} \in \mathcal{A}$ so that each physical configuration appears one time and one time only:

$$\int_{\mathcal{A}} \underbrace{\mathcal{D}\vec{A}\, f_{\mathcal{A}}[\vec{A}]}_{\substack{\text{gauge invariant} \\ \text{under change of } \mathcal{A}}} \Psi^{(a)*}_{\theta^{(i)}}[\vec{A}]\, \Psi^{(b)}_{\theta^{(j)}}[\vec{A}]$$

Note: Under gauge fixed inner product, $\Psi^{(a)}_{\theta^{(i)}}$, $\Psi^{(b)}_{\theta^{(j)}}$ no longer orthogonal for $\theta^{(i)} \neq \theta^{(j)}$

Form of the wave functional & Constraining the Hilbert space

Require: Gauge invariance & $\frac{\delta}{\mathrm{i}\delta\vec{A}(\vec{x})}$ should remain Hermitian under restricted inner product

$$\Longrightarrow \Psi^{(a)}[\vec{A}] \overset{(*)}{=} \Psi^{(a)}[\vec{A}]_{\text{g.i.}} \exp(\mathrm{i}\varphi[\vec{A}])$$

gauge invariant

independent of state (a)

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Now $(\theta - \theta_\Pi) \int \mathrm{d}^3x \,\mathrm{tr}\vec{B} \cdot \frac{\delta}{\mathrm{i}\delta\vec{A}}$ leads to mixing of pure gauge and other directions

$\rightarrow$ Does not allow for above tensor product structure

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Assume for the moment $\theta - \theta_\Pi = 0$

$$H = \frac{1}{2}\int \mathrm{d}^3x \,\mathrm{tr}\left[-g^2\frac{\delta^2}{\delta\vec{A}^2} + \vec{B}^2\right] = -\frac{g^2}{2}\sum_{\sigma\in\sigma_{\text{gauge}}}\frac{\delta^2}{\delta A^2(\sigma)} - \frac{g^2}{2}\sum_{\sigma\in\sigma_{\parallel}}\frac{\delta^2}{\delta A^2(\sigma)} + \frac{1}{2}\int \mathrm{d}^3x \,\mathrm{tr}\,\vec{B}^2$$

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$\longrightarrow$Can find $\Psi^{(a)}[\vec{A}]$ of the above product form

- Only trivial one-dimensional representations of SU(2)
- $U^{(\nu)}$ connected with the identity through transformations nonconstant on boundary

$\Longrightarrow$

- $\Psi^{(a)}[\vec{A}]$ invariant under all symmetry transformations
- in particular $\Psi^{(a)}[\vec{A}_{U^{(1)}}] = \Psi^{(a)}[\vec{A}]$

CP conserved

$$\theta - \theta_\Pi \neq 0$$

Note: $\vec{B}(\vec{x}) = \frac{g}{8\pi^2} \frac{\delta}{\delta \vec{A}(\vec{x})} W[\vec{A}]$

$\longrightarrow$ Redefining derivatives w.r.t. $\vec{A}$ as
$\vec{D}_{\vec{A}} \Psi[\vec{A}] = \mathrm{e}^{\mathrm{i}(\theta-\theta_\Pi) W[\vec{A}]} \frac{\delta}{\delta \vec{A}} \mathrm{e}^{-\mathrm{i}(\theta-\theta_\Pi) W[\vec{A}]} = \mathrm{i} \left(\frac{\delta}{\mathrm{i}\delta \vec{A}} - (\theta - \theta_\Pi) \left(\frac{\delta}{\delta A(\vec{x})} W[\vec{A}] \right) \right) \Psi[\vec{A}]$

- removes cross terms, allows for tensor product form

$$H = \frac{1}{2} \int \mathrm{d}^3x \, \mathrm{tr} \left[g^2 \vec{D}_{\vec{A}}^2 + \vec{B}^2 \right], \quad \int \mathrm{d}^3x \, \vec{D}_{\vec{A}}^2 = \sum_{\sigma \in \sigma_{\text{gauge}}} \vec{D}_{A(\sigma)}^2 + \sum_{\sigma \in \sigma_{\parallel}} \vec{D}_{A(\sigma)}^2$$

- induces translations as

$$T[\Delta \vec{A}] \, \Psi[\vec{A}] = \mathrm{e}^{-\mathrm{i}(\theta-\theta_\Pi)\left(W[\vec{A}+\Delta\vec{A}] - W[\vec{A}] \right)} \Psi[\vec{A} + \Delta \vec{A}]$$

The solutions are invariant under a shift $\Delta \vec{A}_{\text{gauge}}$ using the translation generator $\vec{D}_{\vec{A}}$:

$$T[\Delta \vec{A}_{\text{gauge}}] \, \Psi[\vec{A}] = \Psi[\vec{A}] \quad \text{if} \quad \Psi[\vec{A}] = \mathrm{e}^{\mathrm{i}(\theta-\theta_\Pi) W[\vec{A}]} \Psi_{\text{g.i.}}[\vec{A}]$$

Agrees with reasoning & result in the $\theta - \theta_\Pi = 0$ case

θ in Ψ_θ is pinned to θ in H so that CP is conserved

EFFECTIVE LAGRANGIANS

Spurions and 't Hooft vertex

Chiral anomaly and quark mass matrix M explicitly break $\mathrm{U}(1)_A$
$\rightarrow$ Two associated spurions in the fundamental theory: θ and $\bar{\alpha} = \arg\det M$

DIGA/topology $\rightarrow$ Effective 't Hooft vertex (Γ_{N_f} some coefficient): ['t Hooft (1976,86)]

$$\mathcal{L} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \rightarrow \mathcal{L} - \Gamma_{N_f}\mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{L}\psi_j) - \Gamma_{N_f}\mathrm{e}^{-\mathrm{i}\xi}\prod_{j=1}^{N_f}(\bar{\psi}_j P_\mathrm{R}\psi_j)$$

Summing over topologies (see below) $\rightarrow$

- Primary order of limits $\rightarrow \xi = -\bar{\alpha}$ (no CP violation)
- Exchanging order of limits $\rightarrow \xi = \theta$ (CP violation)

χPT and mass of the η'-meson

χPT as low-energy effective field theory (EFT):

$$U = U_0 e^{\frac{i}{f_\pi}\Phi} \qquad U_0 : \begin{matrix} chiral \\ condensate \end{matrix} \qquad \Phi = \begin{bmatrix} \pi^0 + \eta' & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \eta' \end{bmatrix}$$

$\xi = \theta$: chiral condensate aligned with θ, CP violated

$\xi = -\bar{\alpha}$: chiral condensate aligned with quark mass phases, CP conserved

$$\mathcal{L} = \tfrac{f_\pi^2}{4} \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \tfrac{f_\pi^2 B_0}{2} \mathrm{Tr}(M U + U^\dagger M^\dagger) + |\lambda| e^{-i\xi} f_\pi^4 \det U + |\lambda| e^{i\xi} f_\pi^4 \det U^\dagger \qquad M = \mathrm{diag}\{m_u e^{i\alpha_u}, m_d e^{i\alpha_d}\}$$

- Both spurions of the fundamental theory can appear
  $\rightarrow$ Considering the EFT alone does not lead to a conclusion regarding strong CP
  [contrary to Benabou, Hook, Manzari, Murayama, Safdi 2510.18951]
- The η'-meson is massive due to broken $\mathrm{U}(1)_A$ for either of the above options
  [see also DB Kaplan, Sen 2412.02024]
- In a fixed topological sector, as assumed in 2412.02024, the partition function $\log Z$ depends stricly *linearly* on θ. In that limit, θ acts in the 1PI effective action Γ as a Lagrange multiplier that pins the topological sector. Γ being a well-defined Legendre transform w.r.t. θ relies on convexity of $\log Z[\theta]$, which does not follow unless exchanging the order of limits.
  [cf. Di Vecchia, Sannino, Shore, Veneziano, Yankielowicz 2608.28253]

So is it $\xi = -\bar{\alpha}$ or $\xi = \theta$?—tell from fermion correlations

- Obtain correlation functions from Green's functions in fixed background of instantons and anti-instantons
- Interfere all instanton configurations
  - First, within one topological sector
  - Then over the different sectors

DIGA to dermine CP phase of 't Hooft vertex—not quantitatively accurate for actual QCD

Green's function in n-instanton, $\bar{n}$-anti-instanton background (DIGA)

$$S_{n,\bar{n}}(x,x') \approx S_{0\text{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\hat{\psi}_{0\text{L}}(x-x_{0,\bar{\nu}})\hat{\psi}^{\dagger}_{0\text{L}}(x'-x_{0,\bar{\nu}})}{m e^{-i\alpha}} + \sum_{\nu=1}^{n} \frac{\hat{\psi}_{0\text{R}}(x-x_{0,\nu})\hat{\psi}^{\dagger}_{0\text{R}}(x'-x_{0,\nu})}{m e^{i\alpha}}$$

$\hat{\psi}_{0\text{L,R}}$: 't Hooft zero modes

Comments:

- For small masses, zero modes dominate close to the cores of instantons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Alignment of phase α between Lagrangian mass and instanton-induced χSB $\longrightarrow$ No indication of CP violation here

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cf.

$$S_{0\text{inst}}(x,x') = (-\gamma^{\mu}\partial_{\mu} + m e^{-i\alpha\gamma^5}) \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-x')} \frac{1}{p^2+m^2}$$

(…ntons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]

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Interferences among topological sectors (are immaterial)

Topological quantization $\leftrightarrow$ Interference between sectors for $VT \to \infty$

Fermion correlator

$$\langle \psi(x)\psi^\dagger(x')\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \langle \psi(x)\psi^\dagger(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= S_{0\text{inst}}(x,x') + \kappa \bar{h}(x,x') m^{-1} \mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n\text{)}$$

κ: Number of instantons per unit VT
$\bar{h}(x,x')$: Chiral symmetry breaking tensor

Exchanging the order of limits leads to $\xi = \theta$, based on destructive interference among sectors.

Recall: $S_{0\text{inst}}(x,x') = (-\gamma^\mu \partial_\mu + m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2+m^2}$

$\longrightarrow$ No relative CP-phase between mass and instanton term
$\longrightarrow \xi = -\alpha$
$\longrightarrow CP$ is conserved

Summary

Functional quantization

Primary order of limits: $T \to \infty$ first, then $\sum_{\Delta n} \to CP$ conserved

Canonical quantization

$A^0 = 0$ does not imply $U(\vec{x}) \to$ const. for $|\vec{x}| \to \infty$

$\to$ All $U(\vec{x})$ continuously connected to $\mathbb{1} \to CP$ conserved

EFTs

Do not alone decide question of CP

Summary

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EFTs

Do not alone decide question of CP

THANK YOU!

BACKUP SLIDES

FINITE VOLUME EFFECTS/CONNECTED CONTRIBUTIONS (2512.06827)

Finite volume effects

For large VT, $Z_{\Delta n} \propto I_{\Delta n}(2\kappa VT) \sim \frac{e^{2\kappa VT}}{\sqrt{2\pi 2\kappa VT}}$ in each topological sector when computed based on the primary order of limits.

In [2512.06827], it is stated that the interpretation through exponentiating of connected diagrams "is spoiled by the additional multiplicative factor".

However, the factor is a finite-volume effect that, within $\log Z$, for $\kappa VT \to \infty$ becomes subdominant compared to the exponential and therefore, in that limit, will not contribute to observables.

Connected contributions

To compare with [2512.06827], take the fermion two-point function as ⟨Correlator⟩

$$\langle\psi(x)\bar{\psi}(x')\rangle_{n,\bar{n}} = \frac{1}{n!\bar{n}!}\left[\bar{h}(x,x')\left(\frac{\bar{n}}{m}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}} + \frac{n}{m}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}}\right)(VT)^{\bar{n}+n-1} + S_{\mathrm{free}}(x,x')\,(VT)^{\bar{n}+n}\right](-\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

$$= \langle\psi(x)\bar{\psi}(x')\rangle_1 Z_{n-1,\bar{n}} + \langle\psi(x)\bar{\psi}(x')\rangle_{\bar{1}} Z_{n,\bar{n}-1} + \langle\psi(x)\bar{\psi}(x')\rangle_0 Z_{n,\bar{n}}$$

(the factors Z being interpreted as disconnected vacuum terms) so that we can identify

$$\begin{aligned}
\langle\psi(x)\bar{\psi}(x')\rangle_{\bar{1}} &= \bar{h}(x,x')P_{\mathrm{L}}\,\kappa m^{-1}\mathrm{e}^{-\mathrm{i}\theta}\\
\langle\psi(x)\bar{\psi}(x')\rangle_1 &= \bar{h}(x,x')P_{\mathrm{R}}\,\kappa m^{-1}\mathrm{e}^{\mathrm{i}\theta}\\
\langle\psi(x)\bar{\psi}(x')\rangle_0 &= S_{\mathrm{free}}(x,x')
\end{aligned}$$

The expression with the order of limits exchanged/conventional order

$$\langle\psi(x)\bar{\psi}(x')\rangle = \lim_{VT\to\infty}\lim_{\substack{N\to\infty\\ N\in\mathbb{N}}}\frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N}Z_{\Delta n}} \approx S_{\mathrm{free}}(x,x') + \kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{\mathrm{i}\theta\gamma^5}$$

can then be interpreted as a sum over connected contributions in terms of the correlators with $1, \bar{1}, 0$ subscripts.

Connected contribution–fixed topolgoies

Primary order of limits

$$\langle\psi(x)\bar{\psi}(x')\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N}\langle\psi(x)\bar{\psi}(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}} \approx S_{\text{free}}(x,x') + \kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}$$

This result cannot be viewed as connected in terms of the above correlators.
However, at fixed topology, all correlations are multiplied by the same Δn-dependent phase (cf. previous slide). It is therefore consistent and makes sense to define instead

$$\begin{aligned}
\langle\psi(x)\bar{\psi}(x')\rangle_{\bar{1}} &= \bar{h}(x,x')P_{\mathrm{L}}\,\kappa m^{-1}\mathrm{e}^{\mathrm{i}\alpha} \\
\langle\psi(x)\bar{\psi}(x')\rangle_{1} &= \bar{h}(x,x')P_{\mathrm{R}}\,\kappa m^{-1}\mathrm{e}^{-\mathrm{i}\alpha} \\
\langle\psi(x)\bar{\psi}(x')\rangle_{0} &= S_{\text{free}}(x,x')
\end{aligned}$$

The vacuum diagrams pick up the same global phase, sum to $|Z_{n,\bar{n}}|\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$, and we can construct

$$\langle\psi(x)\bar{\psi}(x')\rangle_{n,\bar{n}} = \left(\langle\psi(x)\bar{\psi}(x')\rangle_{1}|Z_{n-1,\bar{n}}| + \langle\psi(x)\bar{\psi}(x')\rangle_{\bar{1}}|Z_{n,\bar{n}-1}| + \langle\psi(x)\bar{\psi}(x')\rangle_{0}|Z_{n,\bar{n}}|\right)\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

as a result from multiplying correlators with disconnected vacuum terms.
After taking the infinite volume limit, all fixed topology contributions agree (up to phases), so that we can interpret $\langle\psi(x)\bar{\psi}(x')\rangle$ again as sum over connected contributions.

SUBVOLUME EFT

VENEZIANO–DI VECCHIA EFFECTIVE LAGRANGIAN

Finite (sub)volumes

Boundary conditions at infinity crucial for alignment of the CP phases

Calculations in finite spacetime volumes should also be possible for subvolumes of spacetime ($\rightarrow$ open boundary conditions).

Finite vs infinite spacetime volume—cluster decomposition

Consider expectation value of an operator $\mathcal{O}$ in spacetime volume Ω, interfere different topological sectors Δn: [Weinberg QFT]

$$\langle \mathcal{O} \rangle_\Omega = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \frac{\sum_{\Delta n=-N}^{N} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi\, \mathcal{O}\, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-N}^{N} f(\Delta n) \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_\Omega[\phi]}}$$

Factorize into volume contributions, $\Omega = \Omega_1 \cup \Omega_2$: (Assume $\Delta n(\Omega) = \Delta n_1(\Omega_1) + \Delta n_2(\Omega_2)$)

$$\langle \mathcal{O}_1 \rangle_\Omega = \lim_{\substack{N_2\to\infty \\ N_2\in\mathbb{N}}} \lim_{\substack{N_1\to\infty \\ N_1\in\mathbb{N}}} \frac{\sum_{\Delta n_1=-N_1}^{N_1} \sum_{\Delta n_2=-N_2}^{N_2} f(\Delta n_1+\Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-N_1}^{N_1} \sum_{\Delta n_2=-N_2}^{N_2} f(\Delta n_1+\Delta n_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

Independence of $\langle \mathcal{O}_1 \rangle_\Omega$ from the fluctuations in Ω_2 is achieved if the contributions from Ω_2 cancel (absorb determinant phases in f):

$$f(\Delta n_1 + \Delta n_2) = f(\Delta n_1) f(\Delta n_2) \Rightarrow f(\Delta n) = \mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Now keep Δn fixed, Ω_2 either finite or infinite:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2=\Delta n-\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2=\Delta n-\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}$$

$$= \frac{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) I_{\Delta n-\Delta n_1}(2\mathrm{i}\kappa\Omega_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} f(\Delta n) I_{\Delta n-\Delta n_1}(2\mathrm{i}\kappa\Omega_2) \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]}}$$

Subvolume EFT has $\alpha + \theta = 0$

$$\overset{\Omega_2 \gg \Omega_1}{\approx} \frac{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]}}$$

Corresponds to finite VT, $\theta = 0$

Result also holds for $\Omega_2 \to \infty$, Δn free



Path integral makes sense for finite subvolumes with open boundary conditions

The theory with fixed Δn (gauge invariant) in large but finite volumes **complies with cluster decomposition** [cf. Leutwyler, Smilga (1992)]. Lattice results with frozen topology are therefore okay up to finite volume effects.

For $\Omega_2 \to \infty$, the result does not depend on whether Δn is fixed or free.

The Veneziano–Di Vecchia Lagrangian

Effective Lagrangian with $q(x)$ a an auxiliary field: [Veneziano, Di Vecchia (1979)]

$\mathcal{L} = \frac{f_\pi^2}{4} \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \frac{f_\pi^2 B_0}{2} \mathrm{Tr}(M U + U^\dagger M^\dagger) + \frac{1}{2} q(x) \mathrm{Tr}\left[\log U - \log U^\dagger\right] + \frac{N_c}{a f_\pi^2} q^2(x) - \xi q(x)$

Anomalous $\mathrm{U}(1)_A$:

- Identify $q(x)$ with $F\tilde{F}$
- ξ a spurion that restores $\mathrm{U}(1)_A$ invariance
- Again: two available spurions, $\xi = \theta$ and $\xi - \bar{\alpha}$

Assuming a freely fluctuating field $q(x)$ in the full spacetime volume is in conflict with the constraints from topological quantization. An unconstrained $q(x)$ can be considered in an EFT for subvolumes (see previous slides). In that EFT, the coefficient of $q(x)$ is $-\bar{\alpha}$, hence there is no CP violation.

No $\mathrm{U}(1)_A$–CP tension, 1PI-effective action argument from [2608.28253 [hep-th]] does not apply to primary order of limits because the generating functional then is not convex in θ, Legendre transform cannot be carried out (or at most θ acts as a Lagrange multiplier)

TOPOLOGICAL SUSCEPTIBILITY

Topological susceptibility

Topological susceptibility: $\chi = \lim_{k\to 0} \frac{1}{(16\pi^2)^2} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}kx} \langle F\tilde{F}(x) F\tilde{F}(0)\rangle$

Large N analysis: η'-mass proportional to χ from quarkless theory [Witten (1979) Veneziano (1979)]

However, [Albandea, Catumba, Ramos 2402.17518 Benabou, Hook, Manzari, Murayama, Safdi 2510.18951] insist to evaluate χ as

$\chi = \frac{1}{(16\pi^2)^2 \Omega} \int\limits_\Omega \mathrm{d}^4x \int\limits_\Omega \mathrm{d}^4x'\, \mathrm{e}^{\mathrm{i}kx} \langle F\tilde{F}(x) F\tilde{F}(x')\rangle$

i.e. first $k \to 0$, then the d^4x and an extra d^4x' integration

No explanation given in these refs., moreover: Information on propagating particles contained in $\langle F\tilde{F}(x) F\tilde{F}(0)\rangle$, not its double volume integral

Cf. also [Witten (1979)]

Topologic

Topological susceptibility: $\chi = \lim_{k\to 0} \frac{1}{(16\pi^2}$

Large N analysis: η'-mass proportional to $\frac{1}{}$

However, [Albandea, Catumba, Ramos 2402.17518] [Benabou, Hook, Manzari, Murayama, Safdi 2510

$$\chi = \frac{1}{(16\pi^2)^2\Omega} \int_\Omega \mathrm{d}^4x \int_\Omega \mathrm{d}^4x' \, \mathrm{e}^{\mathrm{i}kx} \langle F\tilde{F}(x) F\tilde{F}(x'$$

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No explanation given in these refs., moreov

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Cf. also [Witten (1979)]

In analyzing the formula for $\mathrm{d}^2E/\mathrm{d}\theta^2$ in (6), some subtleties arise. It is useful to introduce an infrared cutoff, which we do by writing

$$\frac{\mathrm{d}^2E}{\mathrm{d}\theta^2} = \frac{1}{N^2}\left(\frac{g^2}{16\pi^2}\right)^2 \lim_{k\to 0} U(k), \tag{7}$$

where

$$U(k) = \int \mathrm{d}^4x \, \mathrm{e}^{ik\cdot x} \langle \mathrm{T}(F\tilde{F}(x) F\tilde{F}(0))\rangle . \tag{8}$$

How does $U(k)$ depend on N? If one computes $U(k)$ in perturbation theory, one finds that it is or order N^2. This is so simply because $U(k)$ receives contributions from each of the N^2 gluon degrees of freedom (fig. 2).



Fig. 2. The two-point function of $F\tilde{F}$ (in one-loop order). It is of order N^2, since any of N^2 gluon species may run around the loop.

However, in perturbation theory, $U(k)$ vanishes at $k = 0$. The reason for this is that $F\tilde{F}$ is a total divergence, so its matrix elements vanish at zero momentum, at least in perturbation theory. It is $U(k)$ at $k = 0$ that controls the θ dependence. As a result, there is no θ dependence in perturbation theory.

While there is no θ dependence in ordinary perturbation theory, there may perfectly well be θ dependence in the $1/N$ expansion. We can see this as follows. Diagram by diagram, $U(k)$ vanishes at $k = 0$. If, however, we first sum the planar diagrams to obtain the large-N limit and then consider $k \to 0$, it is perfectly possible that the limit is non-zero. While individual terms vanish at $k = 0$, the whole sum may not. It is easy to see how this could happen. In perturbation theory the series for $U(k)$ looks like this:

$$U(k) = ak^2 + bg^2k^2 \ln k^2 + cg^4k^2 \ln^2 k^2 + \dots . \tag{9}$$

Rotor analogy

Topological susceptibility: $\chi = \lim_{k\to 0} \frac{1}{(16\pi^2)^2} \int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}kx} \langle F\tilde{F}(x) F\tilde{F}(0)\rangle$

Recent claims that for rotor model, $\chi_{\text{rotor}} = 0$ when using the primary order of limits
Would imply vanishing η'-mass [Albandea, Catumba, Ramos 2402.17518] [Benabou, Hook, Manzari, Murayama, Safdi 2510.18951]

In the analogy, $\frac{1}{16\pi^2} F\tilde{F} \leftrightarrow \frac{1}{2\pi}\dot{\phi}$ for particle of moment of inertia m on circle at angle ϕ

However

For given boundary conditions $\phi(\pm T/2) = \phi_\pm$ and Euclidean time τ:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}\tau'}\langle\phi(\tau)\phi(\tau')\rangle|_{\phi_\pm} = \frac{1}{m}\delta(\tau-\tau') - \frac{1}{Tm} + \left(\frac{\phi_+ - \phi_-}{T}\right)^2$$

Taking $T \to \infty$ first, boundary conditions become irrelevant $\to \chi_{\text{rotor}} = \frac{1}{4\pi^2 m} \neq 0$ [2511.04216]

Back in QCD, no conflict with $\mathrm{U}(1)_A$-breaking, massive η'-meson [see also DB Kaplan, Sen 2412.02024]
Rotor susceptibility only vanishes when taking the $k \to 0$ limit before d^4x integration

THREE-FORM EFT

Fourier transforms of the correlation functions:

$$\widetilde{\langle qq \rangle}(p) = \int \mathrm{d}^4(x-y)\, \mathrm{e}^{\mathrm{i}p(x-y)} \langle q(x) q(y) \rangle = p_\mu p_\nu \widetilde{\langle K_\mu K_\nu \rangle}(p)\,.$$

Connection with topological susceptibility:

$$\lim_{p \to 0} p_\mu p_\nu \widetilde{\langle K_\mu K_\nu \rangle}(p) = \int \mathrm{d}^4 x\, \langle \partial_\mu K_\mu(x) \partial_\nu K_\nu(y) \rangle = \chi\,.$$

The former equation implies e.g. in transverse gauge [Lüscher (1978)], where $\partial_\mu \epsilon_{\mu\nu\rho\sigma} A_\sigma = 0$:

$$\widetilde{\langle K_\mu K_\nu \rangle}(p) = \frac{\chi\, p_\mu p_\nu}{p^4} + \mathcal{O}(p^2)$$

$\rightarrow$ Simple massless pole in $\widetilde{\langle K_\mu K_\nu \rangle}(p)$ for $p_\mu \to 0$.

By definition, K_μ can be understood as the Hodge dual of a three-form field.
$\rightarrow$ Compare it with the action for a massless three-form L_μ,

$$S = \int_\Omega \mathrm{d}^4x \, (\partial_\mu L_\mu)^2 \, .$$

"Massless" refers here to the absence of a mass term in the action and is not supposed to indicate that there is a massless propagating degree of freedom, which in fact there is not.

Variation: $$\delta S = -2 \int_\Omega \mathrm{d}^4x \, (\partial_\nu \partial_\mu L_\mu) \, \delta L_\nu + 2 \int_{\partial\Omega} \mathrm{d}a_\nu \, (\partial_\mu L_\mu) \, \delta L_\nu \, ,$$

where a_ν is a normal surface element on Ω.

Discarding the boundary term, one would conclude that $\partial_\mu L_\mu = \text{const}$. However, dropping boundary term not justified because it in general contributes to the equations of motion.

The situation is therefore different from QCD in an infinite spacetime volume Ω: There, solutions of finite action exist while the topological term does not need to vanish and can be written as a boundary term. However, within a given topological sector the boundary term gives a fixed contribution $-\mathrm{i}\theta\Delta n$ to the action. Therefore it has no impact on the equations of motion.

Three-form theory solutions

Back to the three-form action, we must therefore make further assumptions about the boundary term. While there are many different options, we follow [Dvali 1705.06317] that works with solutions

$$\partial_\mu L_\mu = \theta_L \,,$$

where θ_L is an integration constant. This appears to require boundary conditions $\delta L_\mu = 0$ on $\partial\Omega$ and

$$\theta_L = \frac{1}{\Omega}\int_{\partial\Omega} \mathrm{d}a_\nu \, L_\nu \,.$$

Interpreting L_μ as a four-dimensional electric field in the background of a constant charge density θ_L,we see that $\partial_\mu L_\mu = \theta_L$ has solutions that satisfy these boundary conditions. Note that for a given θ_L, $L_\mu \sim \theta_L \Omega^{1/4}$. To keep the boundary conditions well-defined in terms of finite L_ν, we therefore have to impose these on finite Ω or we have to take $\theta_L = 0$. Given the boundary condition $\delta L_\mu = 0$, can also derive EOMs for the correlators:

$$\partial^x_\mu \partial^x_\nu \langle L_\nu(x) L_\rho(y)\rangle = -\delta_{\mu\rho}\delta^4(x-y) \quad \Rightarrow \quad \langle (\partial_\mu L_\mu(x))(\partial_\nu L_\nu(y))\rangle = \delta^4(x-y) + \text{const.}$$

Note the similarity with the rotor susceptibility.

While there are similarities with QCD, we therefore see that θ_L is a boundary condition on a *finite* volume. In this sense it is similar to the winding number Δn that is implied by the pure gauge boundary condition on a fixed topological sector, and not θ that does not select boundary conditions in first place.

Footnote from 2209.14219

[2] The 3-form language of [14] clarifies the claim of [24] that by changing the order of limits in ordinary instanton calculation, one ends up with $\vartheta = 0$. In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in (28) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is $E_0 = 0$ which is equivalent to $\vartheta = 0$. Other states $E \neq 0$ (corresponding to $\vartheta \neq 0$) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the $\vartheta \neq 0$ vacua are of course present but one is constrained to $\vartheta = 0$ by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the ϑ-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with $\vartheta \neq 0$ ($E \neq 0$), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.

- Exchanged/conventional order of limits ($\sum_{\Delta n}$ first, then $VT \to \infty$) predicts CP violation with IR cutoff, contrary to the claimed projection on E_0
- If finite volume is problematic, more reason to take the infinite volume limit as soon as possible, as we do, leading to no CP violation for arbitrary θ

ALL ON FERMION CORRELATIONS

So is it $\xi = -\bar{\alpha}$ or $\xi = \theta$?—tell from fermion correlations

The effective vertex generates the following correlation functions at tree level:

$$\langle \prod_{j=1}^{N_f} \psi_j(x_j) \bar{\psi}_j(x_j') \rangle_{\text{inst}} = \left(\mathrm{e}^{-\mathrm{i}\xi} \prod_{j=1}^{N_f} P_{\mathrm{L}j} + \mathrm{e}^{\mathrm{i}\xi} \prod_{j=1}^{N_f} P_{\mathrm{R}j} \right) \bar{H}(x_1, \ldots, x_1', \ldots)$$

Goal: Compute correlation function and compare with EFT answer above to fix ξ

Cf. leading contribution to two-point function

$$\langle \psi_i(x) \bar{\psi}_j(x') \rangle = \mathrm{i} S_{0\text{inst}\,ij}(x, x')$$

$$\mathrm{i} S_{0\text{inst}\,ij}(x, x') = (-\gamma^\mu \partial_\mu + \mathrm{i} m_i \mathrm{e}^{-\mathrm{i}\alpha_i \gamma^5}) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{\delta_{ij}}{p^2 - m_i^2 + \mathrm{i}\epsilon}$$

So $\xi = \theta / \xi = -\bar{\alpha}$ implies CP-violation/no CP-violation

Only one explicit calculation based on dilute instanton gas (DIGA) finding $\xi = \theta$
['t Hooft (1986)]

Fermion correlations

- Obtain correlation functions from Green's functions in fixed background of instantons and anti-instantons
- Interfere all instanton configurations
  - First, within one topological sector
  - Then over the different sectors

DIGA to dermine CP phase of 't Hooft vertex—not quantitatively accurate for actual QCD

Green's function in n-instanton, $\bar{n}$-anti-instanton background (DIGA)

$$S_{n,\bar{n}}(x,x') \approx S_{0\text{inst}}(x,x') + \sum_{\bar{\nu}=1}^{\bar{n}} \frac{\hat{\psi}_{0\text{L}}(x-x_{0,\bar{\nu}})\hat{\psi}^{\dagger}_{0\text{L}}(x'-x_{0,\bar{\nu}})}{m e^{-i\alpha}} + \sum_{\nu=1}^{n} \frac{\hat{\psi}_{0\text{R}}(x-x_{0,\nu})\hat{\psi}^{\dagger}_{0\text{R}}(x'-x_{0,\nu})}{m e^{i\alpha}}$$

$\hat{\psi}_{0\text{L,R}}$: 't Hooft zero modes

Comments:

- For small masses, zero modes dominate close to the cores of instantons, far away from instantons the solution goes to the zero-instanton configuration [Diakonov, Petrov (1986)]
- Alignment of phase α between Lagrangian mass and instanton-induced χSB $\longrightarrow$ No indication of CP violation here

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cf.
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Interferences within the topological sectors

Within a topological sector, interfere/sum/integrate over

- all instanton/anti-instanton numbers $n + \bar{n}$ with $\Delta n = n - \bar{n}$ fixed
- locations of all instantons/anti-instantons
- remaining collective coordinates

$\longrightarrow$ Dilute instanton gas approximation (skip technicalities)

Can also obtain coincident fermion correlations using the index theorem and anomalous current only

Correlation function for fixed Δn

$$\langle\psi(x)\bar\psi(x')\rangle_{\Delta n} = \sum_{\substack{\bar n, n\geq 0 \\ n-\bar n=\Delta n}} \frac{1}{\bar n! n!}\left[\bar h(x,x')\left(\frac{\bar n}{m\mathrm{e}^{-\mathrm{i}\alpha}}P_{\mathrm{L}} + \frac{n}{m\mathrm{e}^{\mathrm{i}\alpha}}P_{\mathrm{R}}\right)(VT)^{\bar n+n-1} + S_{\mathrm{free}}(x,x')\,(VT)^{\bar n+n}\right] \times \kappa^{\bar n+n}(-1)^{n+\bar n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

$$= \left[\left(\mathrm{e}^{\mathrm{i}\alpha}I_{\Delta n+1}(2\kappa VT)P_{\mathrm{L}} + \mathrm{e}^{-\mathrm{i}\alpha}I_{\Delta n-1}(2\kappa VT)P_{\mathrm{R}}\right)\frac{\kappa}{m}\,\bar h(x,x') + I_{\Delta n}(2\kappa VT)S_{\mathrm{free}}(x,x')\right] \times (-1)^{\Delta n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$

Instantons per spacetime volume: $\kappa \propto \mathrm{e}^{-S_{\mathrm{E}}}$

χSB rank-two spinor-tensor from integrating quark zero-modes over the locations of the instantons: $\bar h(x,x')$

Modified Bessel function: $I_\nu(x)$

Sum is dominated by particular value of $n \approx \bar n$: [Diakonov, Petrov (1986)]

$$\langle n\rangle = \frac{\sum_{n=0}^{\infty} n\frac{(\kappa VT)^n}{n!}}{\sum_{n=0}^{\infty}\frac{(\kappa VT)^n}{n!}} = \kappa VT\,, \qquad \frac{\sqrt{\langle(n-\langle n\rangle)^2\rangle}}{\langle n\rangle} = \frac{1}{\sqrt{\kappa VT}}\,, \qquad \text{cf. } \lim_{x\to\infty}\frac{I_{\Delta n}(x)}{I_{\Delta n'}(x)} = 1$$

$\longrightarrow$ No relative CP phase between mass and instanton induced breaking of χral symmetry—alignment in infinite-volume limit

Correspondingly, partition function for fixed Δn: [cf. Leutwyler, Smilga (1992)]

$$Z_{\Delta n} = I_{\Delta n}(2\kappa VT)\,(-1)^{\Delta n} e^{i\Delta n(\alpha+\theta)}$$

There is no CP violation/misalignment of phases until this point. It remains to consider the interference between the topological sectors.

Interferences among topological sectors (are immaterial)

Topological quantization $\leftrightarrow$ Interference between sectors for $VT \to \infty$

Fermion correlator

$$\langle \psi(x)\psi^\dagger(x')\rangle = \lim_{\substack{N\to\infty \\ N\in\mathbb{N}}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} \langle \psi(x)\psi^\dagger(x')\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} Z_{\Delta n}}$$

$$= S_{0\text{inst}}(x,x') + \kappa \bar{h}(x,x') m^{-1} \mathrm{e}^{-\mathrm{i}\alpha\gamma^5} \quad \text{(same as for fixed } \Delta n\text{)}$$

Recall: $S_{0\text{inst}}(x,x') = \left(-\gamma^\mu \partial_\mu + m\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}p(x-x')} \frac{1}{p^2+m^2}$

$\longrightarrow$ No relative CP-phase between mass and instanton term

$\longrightarrow \xi = -\alpha$

$\longrightarrow CP$ is conserved

Limits ordered the other way around

First sum over all Δn as well:

$$\sum_{\bar{n},n\geq 0}\frac{1}{\bar{n}!n!}\Big[\bar{h}(x,x')(\bar{n}\,m^{-1}\mathrm{e}^{\mathrm{i}\alpha}P_{\mathrm{L}}+n\,m^{-1}\mathrm{e}^{-\mathrm{i}\alpha}P_{\mathrm{R}})\,(VT)^{\bar{n}+n-1}+S_{0\mathrm{inst}}(x,x')\,(VT)^{\bar{n}+n}\Big]$$
$$\times\,(-m\kappa)^{\bar{n}+n}\mathrm{e}^{\mathrm{i}\Delta n(\alpha+\theta)}$$
$$=\Big[-\left(\mathrm{e}^{-\mathrm{i}\theta}P_{\mathrm{L}}+\mathrm{e}^{\mathrm{i}\theta}P_{\mathrm{R}}\right)\frac{\kappa}{m}\bar{h}(x,x')+S_{0\mathrm{inst}}(x,x')\Big]\,\mathrm{e}^{-2\kappa\,VT\cos(\alpha+\theta)}$$

$$Z\to\sum_{n,\bar{n}}\frac{1}{n!\bar{n}!}(-\kappa\,VT)^{\bar{n}+n}\mathrm{e}^{-\mathrm{i}(\bar{n}-n)(\alpha+\theta)}=\mathrm{e}^{-2\kappa\,VT\cos(\alpha+\theta)}$$

Then, $VT\to\infty$ trivial as VT-dependence cancels
$\longrightarrow$ Relative CP phase leading to CP-violating observables

However: Changing the order does not correspond to a nonsingular integration contour.