Limit cycles

in a two dimensional monetary policy model



Keynesian model with inflation targeting

- State variables: *interest rate* r and *inflation of prices* π
- *time adjustment constants* β, γ, δ
- *inflation target* set by the central bank $\bar{\pi}$
- *normal output* $\bar{Y}$ and *Allee-type threshold* $\bar{Y}_n \equiv w \times \bar{Y}$
- *lower bound of the output* $B \equiv p \times \bar{Y}$
- output at *balance* $r = \pi$: $B + k$
- *S-type model* for output as function of $r - \pi$

$$Y = B + \frac{k}{e^{r-\pi}}$$

- *Dynamical system*:

$$\dot{r} = (\pi - \bar{\pi}) + \beta(Y - \bar{Y}) \tag{1}$$

$$\dot{\pi} = \gamma(\bar{\pi} - \pi)\frac{Y}{Y + Y_n} + \delta(Y - \bar{Y}) \tag{2}$$

Normal equilibrium and Hopf bifurcation

- *Normal* ($\pi = \bar{\pi}$) *equilibrium* (NE) is the only stationary point for $0 < p < 1$

$$r_e = \ln Q,\ \pi_e = \bar{\pi}, \quad k = Q \times \frac{1-p}{e^{\bar{\pi}}}\bar{Y}$$

- Linearization at NE is traceless iff

$$\gamma = \gamma_0 = (1+w)(1-p)(\delta - \beta)\bar{Y}$$

- Threshold γ_0 is positive iff $\beta = q \times \delta$, where $0 < q < 1$
- NE is focus or centre at γ_0 provided $\gamma_0 > 0$ with related circular frequency

$$\omega_0 = \sqrt{(1-p)\xi[1 + (1-p)(1-q)q\xi]}$$

with non-negative parameter

$$\xi = \delta\bar{Y}$$

Theorem 0

The system (1)-(2) undergoes Hopf bifurcation at NE at $\gamma = \gamma_0$ for every $0 < p, q < 1$ and every $w \geq 0$, and NE is linearly unstable for $\gamma < \gamma_0$ and it is linearly stable for $\gamma > \gamma_0$

Direction and degeneracy of Hopf bifurcation

Translation into origin $(r, \pi) \mapsto (x + r_e, y + \pi_e)$

$$\dot{x} = y - (1-p)q\xi(1 - e^{y-x}) \tag{3}$$

$$\dot{y} = -(1-p)\xi(1 - e^{y-x}) - \frac{(1-p)e^y + pe^x}{(1-p)e^y + (p+w)e^x}\gamma y \tag{4}$$

Polar form of Hopf bifurcation

$$\dot{\rho} = \rho \times \left\{-\frac{\gamma - \gamma_0}{1+w} + l_1\rho^2\right\}$$

with the first Lyapunov quantity

$$l_1 = \mathcal{L}_1 \times \hat{l}_1$$

$\mathcal{L}_1$ is non-negative

$$\mathcal{L}_1 = \frac{\xi(1-p)^3(1-q)[1 + \xi(1-p)q]}{8(1+w)^2[1 + \xi(1-p)(1-q)q]} \times w$$

and $\hat{l}_1$ is a linear function of w:

$$\hat{l}_1 = \beth_0 + \beth_1 w$$

where

$$\beth_0 = 1 + \xi(1-q)\times [3 - p(3+q) + \xi(1-p)(1-q)(2-3p)q]$$

$$\beth_1 = -\xi(1-q)[1 - p(1-p) + \xi(1-p)^2pq]$$

and $\beth_1$ is negative for every admissible (p, q, ξ)

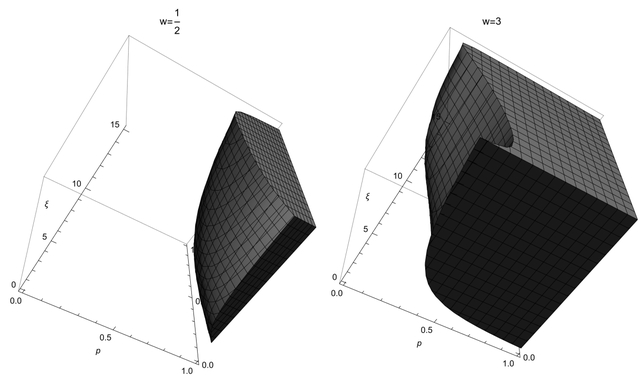


Figure 1. Supercritical (emphasized) vs subcritical Hopf bifurcation by means of the sign of $\hat{l}_1$

Bautin's point (NE, γ_0, w_B):

$$w_B = -\frac{\beth_0}{\beth_1} \xrightarrow{\xi \to +\infty} \frac{(2-3p)(1-p)}{(1-q)p}$$

is asymptotically (for large economies) admissible only for $p < \frac{2}{3}$

Theorem 1

The system (1)-(2) undergoes non-degenerate Hopf bifurcation at NE at $\gamma = \gamma_0$ provided

$$w \neq 0 \ \wedge\ w \neq w_B$$

and this bifurcation tends to be supercritical for every admissible (p, q, ξ) and every w large enough

Degenerated case $w = 0$

For $w = 0$ and $\gamma = \gamma_0$ the system (3)-(4) reads for $z = y - x$:

$$\dot{z} = u \tag{5}$$

$$\dot{u} = -\gamma_0(1 - e^z)\left(u - \frac{\omega_0^2}{\gamma_0}\right), \tag{6}$$

The system (5)-(6) possesses the first integral

$$(z, u) \mapsto -\frac{u}{\gamma_0} - \ln|\gamma_0 u - \omega_0^2| + e^z - z$$

and, by Theorem 0, NE is a focus at $\gamma = \gamma_0$, neutral or not

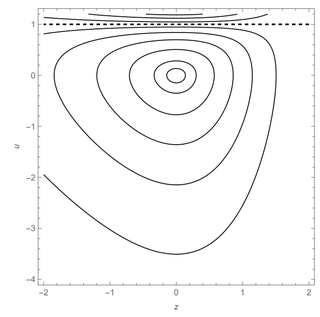


Figure 2. Contours of V in the vicinity of NE

Theorem 2

In the case $w = 0$ the NE is non-linear centre at $\gamma = \gamma_0$ (all Lyapunov quantities vanish)

References

[1] T. Asada.
Modeling financial instability.
European Journal of Economics and Economic Policies, 9:215 – 232, 2012.

[2] M. Demetrian T. Asada and R. Zimka.
On dynamics in a keynesian model of monetary and stabilization policy with debt effect.
Communications in Non-linear Science and Numerical Simulations,, 58:131 –146, 2018.

This work was supported by the VEGA programme under Contract no. 1/0084/23 and 1/0346/25

FAKULTA MATEMATIKY, FYZIKY A INFORMATIKY
Univerzita Komenského v Bratislave

MATFYZ CONNECTIONS

Michal Demetrian
Comenius University in Bratislava

Rudolf Zimka, Emília Zimková
Matej Bel University in Banská Bystrica